ON THE DYNAMICS OF A SECOND ORDER NONLINEAR DIFFERENCE EQUATION

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ABSTRACT

ON THE DYNAMICS OF A SECOND ORDER NONLINEAR DIFFERENCE EQUATION

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In this thesis, a certain second order fractional difference equation containing two arbitrary parameters is handled. The issue equation is investigated with aspects of some dynamics structures: the boundedness character and semi-cycle analysis of positive solutions are examined; existence of periodic solutions is studied; local and global stability analysis of the fixed point are performed.

This thesis consists of four chapters. In the first chapter, historical information about difference equations, some modelings with them, and some recent studies are given. In the second chapter, basic concepts and known results concerning the sequences and difference equations are provided. Main results are presented in Chapter 3. A short conclusion is written down in the last chapter.

Keywords: Difference equations, boundedness, stability, oscillation, periodicity

ÖZ

İKİNCİ MERTEBEDEN LİNEER OLMAYAN BİR FARK DENKLEMİNİN DİNAMİKLERİ ÜZERİNE

Aksoy, Aycan

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Bu tezde iki keyfi parametre içeren ikinci dereceden özel bir rasyonel fark denklemi ele alınmıştır. Bu denklem bazı dinamik yapıları incelenmiştir: pozitif çözümlerin kararlılık ve yarı döngü analizleri; periyodik çözümlerin varlığı; denge noktasının yerel ve global kararlılık analizleri yapılmıştır.

Bu tez dört bölümden oluşmaktadır. İlk bölümde fark denklemleri hakkında tarihsel bilgi, bunların bazı modellemeleri, ve yakın zamanda yapılmış bazı çalışmalar verilmiştir. İkinci bölümde, diziler ve fark denklemleriyle ilgili bilinen tanımlar ve sonuçlar gösterilmiştir. Asıl sonuçlar Bölüm 3'te sunulmuştur. Son bölümde kısa bir sonuç yazılmıştır.

Anahtar Kelimeler: Fark denklemleri, sınırlılık, kararlılık, salınımlık, periyodiklik

To my family

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INTRODUCTION

Mathematical computations often are based on equations. Some equations make problems possible for finding a way to separate them into sub-problems which are in the same form of the original problems. If this process of separating is repeated many times, the last sub-problem is small and easily solved. Also, the solutions of the original problems are found easier thanks to the solutions of the sub-problems which are called “difference equation” or “recurrence equation.”

Many real world processes are studied by means of difference equations. Because of their wide range of applications in mechanics, economics, electronics, chemistry, ecology, biology, etc., the theory of discrete dynamical systems has been under intensive development and many researchers have been paying their attention to the study of these systems.

In this thesis, a certain second order nonlinear difference equation containing two arbitrary parameters shall be considered.

This thesis has been organized as follows:

The present chapter focuses on explaining the emergence of difference equations and illustrating applications and diversities of these equations. Starting from a short historical background about difference equations, some primeval types are shown and the rise of varieties of them is illustrated in the first part. In Section 1.2, some application areas of difference equations are mentioned, some famous models are indicated, and also some examples are shared to show how a difference equation is formulated. A selection of recent studies of some researchers are given in the last section.

Chapter 2 is divided into two sections; sequences and difference equations, respectively. Basic definitions and some known results are given in these sections. Second order autonomous difference equations are explained as a subsection of difference equations because the given results in that chapter are useful in main results.

Chapter 3 consists of six sections. In the first section a second order difference equation is defined, then the next section shows some lemmas and corollaries which results are necessary for other sections. Section 3.3 is related only positive solutions. Periodicity and semi-cycle analysis are given in Section 3.4. However, semi-cycle analysis is done for the positive solutions of defined equation. After this section, stability of equilibrium point of this equation is analyzed. This chapter is finished with numerical examples.

Finally, the last chapter is devoted to a conclusion.

1.1 History of Difference Equations

The notions of computing by recursion exists for a long time, like counting. It dates back to 2000 B.C. when Babylonians aimed to compute the roots. In an effort to extract roots, Babylonians put forward the primitive form of a difference equation. However, more extensive form of this arose about 450 B.C. in the Pythagoreans' study of figurative numbers, like the triangular numbers which satisfy the difference equation in modern notation

$$t_n = t_{n-1} + n,$$

the square numbers satisfying the equation

$$s_n = s_{n-1} + n^2,$$

and so on.

Also, the system of difference equations

$$x_n = x_{n-1} + 2y_{n-1},$$

$$y_n = x_{n-1} + y_{n-1}$$

was used by Pythagoreans to generate large solutions of Pell's equation,

$$x^2 - 2y^2 = 1,$$

and so approximations of $\sqrt{2}$ had been calculated. Around 250 B.C., while trying to compute the circumference of a circle, Archimedes found out equations of the form

$$P_{2n} = 2p_n P_n/(p_n + P_n), \quad p_{2n} = \sqrt{p_n P_{2n}}$$

to calculate the perimeters P_n and p_n of the circumscribed polygon of n sides and the inscribed polygon of n sides, respectively.

At the beginning of the thirteenth century Fibonacci formulated the following well-known sequence

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

which was transformed into a difference equation, in 1634, by Albert Girard

$$F_n = F_{n-1} + F_{n-2}.$$

The last equation was solved by de Moivre in 1730.

In 1572 Bombelli studied the equation

$$y_n = 2 + 1/y_{n-1}$$

which is akin to the equation

$$z_n = 1 + 1/z_{n-1}$$

provided by ratios of Fibonacci numbers so as to approximate $\sqrt{2}$. While Fibonacci gave a sketchy definition for the view of continued fractions, which are nearly related with difference equations, Cataldi discovered a more accurate definition around 1613.

Chia Hsien (*around* 1050) and Omar Khayyam (*around* 1100) deciphered the first known example of a difference equation in two indexes that is the equation

$$b_{n+1,r} = b_{n,r} + b_{n,r-1}$$

for the binomial coefficients.

In the sixteenth century, the method of recursion was crucially progressed thanks to the invention of mathematical induction by Francesco Maurolico and with its development by Fermat and Pascal in the seventeenth century.

The mathematical models used in astronomy and physics were continuous and so the infinitesimal calculus was constructed as the natural tool for the analysis of some discrete models. However, the models were inadequate especially in probability theory that's why a new type of calculus was needed. Hence, the calculus of finite differences were found by Sir Thomas Harriet (1560 – 1621), and it was applied to the calculation of logarithms by Henry Briggs (1556 – 1630). Leibniz resolved it around 1672. This calculus was used to study interpolation theory by Newton, Euler, Lagrange, Gauss, and many others. At the beginning of the eighteenth century, the theory of finite differences put forward by Stirling. At the same time, a significant category of nonlinear difference equations, which is known as Newton's method in the present day, was used by Newton around 1669 to study solutions of $y^3 - 2y - 5 = 0$ and then in computations for Kepler's equation. In 1690, Raphson worked out a more organized implementation of the method. Next crucial member of nonlinear difference equations consists of pairs of equations involving arithmetic and geometric means. For example, Lagrange obtained the equations

$$x_n = (x_{n-1} + y_{n-1})/2, \quad y_n = \sqrt{x_{n-1}\, y_{n-1}}$$

as an algorithm for the reduction and evaluation of elliptic integrals. Gauss and Borhardt discovered related algorithms, and Gauss' researches led him to the discovery of elliptic functions.

The fundamental theory of linear difference equations was expanded in the eighteenth century by de Moivre, Euler, Lagrange, Laplace. Laplace utilized generating functions as part of his work in probability theory, which were used firstly by de Moivre to solve the Fibonacci equation. Poincaré established the first steps in studying the asymptotic properties of solutions of linear difference equations in the 1880's. The general idea of asymptotic series is formulated by Poincaré. Also, he illustrated that the ratio of consecutive values of a solution must converge to a characteristic root in appropriate circumstances. An important development of this result was found by Perron in 1990.

In the course of the 1950's, simple nonlinear difference equations, with the inclusion of logistic equation, were used by some different ecologists to study modifying populations from one season to another. They put emphasis on the stability of the iteration.

However, in the early 1970's May investigated the variety of complex behavior exhibited by the logistic equation and pondered the possible relationship of this behavior to observed fluctuations in real populations. Additional discoveries about the logistic and related equations were soon made by York, Sarkovskii, Feigenbaum, and others, and the significantly intricate properties of these equations led to their becoming a focus in the developing area of chaotic dynamical systems. The excitement of these discoveries attracted the attention of researchers who attempted to apply the results to various fields from economics to medicine.

1.2 Some Modelings with Difference Equations

Difference equations have been used to model a wide variety of forms both in mathematics itself and in its applications in different fields such as economics, computing, electrical circuit analysis, population dynamics, biology, and other fields.

Some famous models have been illustrated in the context. These models can also be found in [11, 14, 27, 28, 30]

(i) Population growth (Malthusian) model:

$$x_{n+1} = ax_n$$

where a is the growth rate.

(ii) Logistic growth model:

$$x_{n+1} = x_n[(1 + a) - bx_n]$$

where b is the competitional date rate.

(iii) Prey-predator model:

$$x_{n+1} - x_n = -ax_n + bx_n y_n$$
$$y_{n+1} - y_n = cy_n - dx_n y_n$$

where $a, b, c, d \geq 0$.

(iv) Competition model:

$$x_{n+1} - x_n = ax_n - bx_n y_n$$
$$y_{n+1} - y_n = cx_n - dx_n y_n$$

where $a, b, c, d \geq 0$.

(v) Contagious disease model:

$$x_{n+1} - x_n = -\beta x_n y_n$$
$$y_{n+1} - y_n = \beta x_n y_n$$

where $\beta > 0$.

(vi) The classical Hansen-Samuelson's accelerator-multiplier model:

$$Y_n = c\,Y_{n-1} + \alpha(Y_{n-1} - Y_{n-2}) + A_0$$

where $A_0 = C_0 + I_0 + G_0$ is a constant which is represents the sum of the minimum consumption, Y_n is the national income in period n, $I_n = \alpha(Y_{n-1} - Y_{n-2})$ is the net investment in period n, $\alpha > 0$ is the accelerator, and $c \in (0, 1)$.

(vii) The equation

$$I_{n+1} - (2 + \frac{R_1}{R_2})I_n + I_{n-1} = 0$$

is used to find the current in the n^{th} loop of an electrical network consisting of resistors and a voltage source. In this equation R_1 and R_2 are constant resistances and I_n denotes the current in the n^{th} loop.

In what follows some examples are provided to illustrate how difference equations can be formulated. These examples can be found in [14, 22, 31]

Example 1.2.1 *It is observed that the decrease in the mass of a radioactive substance over a fixed time period is proportional to the mass that was present at the beginning of the time period. If the half life of radium is* 1600 *years, find a formula for its mass as a function of time.*

Denoting the mass of the radium after t years by $m(t)$, *one gets*

$$m(1) - m(0) = -km(0),$$

from which $m(1) = (1 - k)m(0)$ *is derived. Similarly,*

$$\begin{aligned} m(2) &= (1-k)m(1) = (1-k)^2 m(0), \\ m(3) &= (1-k)m(2) = (1-k)^3 m(0), \\ &\vdots \end{aligned}$$

Continuing in this way one obtains the difference equation

$$m(t) - m(t-1) = -km(t-1) \quad \text{for} \quad t = 1, 2, \ldots$$

By mathematical induction $m(t)$ *is found as*

$$m(t) = (1-k)^t m(0).$$

Since after 1600 *years, half of the initial amount is remaining, one writes*

$$\frac{1}{2}m(0) = m(1600) = m(0)(1-k)^{1600}$$

yielding $(1 - k) = 2^{-1/1600}$. *Thus,*

$$m(t) = m(0)2^{-t/1600}.$$

Example 1.2.2 *(The Tower of Hanoi Problem) In* 1883 *Édouard Lucas, a French mathematician invented a puzzle that he called Tower of Hanoi. The puzzle consisted of three pegs, and there were* 8 *wooden disks piled in the order of decreasing size on one peg. The aim in this puzzle was to move all the disks one by one from one peg to another never placing a larger disk on top of a smaller one. What is the minimum number of moves to solve the problem?*

Call the pegs A,B, and C. Suppose that all the $n + 1$ *disks are initially on the peg A. Suppose also that* n *disks can be moved from one peg to another in* y_n *moves. Then to move all the* $n + 1$ *disks from peg A to another peg, say C, obeying the stated rules, one can move the top* n *disks from peg A to peg B. Then the largest disk remaining on the peg A can be transferred from peg A to peg C in one move which when followed by the* y_n *moves required to move the* n *disks from peg B to peg C solves the puzzle. Therefore,* y_n *must satisfy the recurrence relation*

$$y_{n+1} = y_n + 1 + y_n = 2y_n + 1.$$

The solution satisfying the initial condition $y(1) = 1$ *can be found to be*

$$y_n = 2^n - 1, \quad n = 1, 2, \ldots .$$

Example 1.2.3 *(Airy equation) Suppose we wish to solve the differential equation*

$$y''(x) = x\, y(x).$$

The Airy equation appears in many calculations in applied mathematics. For example, in the study of nearly discontinuous periodic flow of electric current and in the description of the motion of particles governed by the Scródinger equation in quantum mechanics. One approach is to seek power series solutions of the form

$$y(x) = \sum_{k=0}^{\infty} a_k\, x^k.$$

Substitution of the series into the differential equation yields

$$\sum_{k=2}^{\infty} a_k\, k(k-1)x^{k-2} = \sum_{k=0}^{\infty} a_k\, x^{k+1}.$$

The shift of index in the series on the left side of the equation gives us

$$\sum_{k=-1}^{\infty} a_{k+3}(k+3)(k+2)\, x^{k+1} = \sum_{k=0}^{\infty} a_k\, x^{k+1}.$$

These series are equal on an interval of x values, where they both converge, if and only if

$$a_{k+3}(k+3)(k+2) = a_k,$$

which gives us

$$a_{k+3} = \frac{a_k}{(k+3)(k+2)}.$$

The last equation is a difference equation that allows us to compute (in principle) all coefficients a_k*, and hence the solution* $y(x)$ *in terms of the coefficients* a_0 *and* a_1*.*

Example 1.2.4 *Euler's method for approximating the solution of the initial value problem*

$$x'(t) = f(t, x(t)), \qquad x(t_0) = x_0,$$

is obtained by replacing $x'(t)$ with the difference quotient $\frac{x(t+h)-x(t)}{h}$ for some small displacement h. We have

$$\frac{x(t+h)-x(t)}{h} = f(t, x(t))$$

or

$$x(t+h) = x(t) + hf(t, x(t)).$$

To change this difference equation to a more conventional form, let $x_n = x(t_0 + nh)$ for $n = 0, 1, 2, \ldots$. Then

$$x_{n+1} = x_n + hf(t_0 + nh, x_n) \qquad n = 0, 1, 2, \ldots ,$$

where x_0 is given. The approximating values x_n can now be computed recursively; although, the approximations may be useful only for restricted values of n. This is an example of nonlinear difference equation.

Example 1.2.5 *(The $3x+1$ problem) Another example of a nonlinear difference equations is*

$$x_{n+1} = \begin{cases} x_n/2 & \text{if } x_n \text{ is even} \\ (3x_n+1)/2 & \text{if } x_n \text{ is odd} \end{cases}$$

for $n \geqslant 0$, where x_0 chosen to be a positive integer so that every element in the sequence is a positive integer. Although the two-part description given above is the simplest one, we can also write the difference equation as a single expression

$$x_{n+1} = x_n + \frac{1}{4} - \frac{2x_n+1}{4}\cos(\pi x_n).$$

To investigate the behavior of solutions of the difference equation, let's try a starting value of $x_0 = 23$. The solution sequence is

$$\{x_n\} = \{23, 35, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1, 2, 1, \ldots\}.$$

Note that once the sequence reaches the value 2, it alternates between 2 and 1 from that point on. The famous $3x+1$ problem (also called Collatz Problem) asks whether every starting positive integer eventually results in this alternating sequence. As of this writing, it is known that every starting integer between 1 and 5.6×10^{13} does lead to alternating sequence, but no one has been able to solve the problem in general.

Let us now move on to more recent developments which motivated us in preparing the current thesis.

1.3 Some Studies

In Section 1.1, the ideas of recursion, primitive forms of difference equations, inventors and developers of some special types of difference equations are illustrated. Nowadays, many researchers have been still continuing to examine some special difference equations with aspects of stability, oscillatory, boundedness character, and periodicity. In this section, a selection of recent studies of some researchers are given with different aspects. For a detailed analysis of the following difference equations, see [1, 3, 4, 12, 19, 20, 25]

In 1994, S.A Kuruklis studied the $(k+1)$th order difference equation, which is homogeneous with constant coefficient,

$$x_{n+1} - ax_n + bx_{n-k} = 0 \quad n = 0, 1, 2, \dots \tag{1.1}$$

where the coefficient a and b are real numbers and k is a non-negative integer. He examined the asymptotic stability of the difference equation (1.1).

In 1999, A.M. Amleh, E.A. Grove, and G. Ladas analyzed the global stability, the boundedness character, and the periodic nature of the positive solutions of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \dots, \tag{1.2}$$

where $\alpha \in [0, \infty)$, and the initial conditions x_{-1} and x_0 are arbitrary positive real numbers. In 2006, A.E. Hamza investigated the global stability, the permanence, and the oscillation character of the recursive equation (1.2) for nonnegative values of the parameter α with negative initial conditions x_{-1} and x_0.

R.M. Abu-Saris and R. DeVault, in 2003, examined the global stability of the nonlinear difference equation

$$y_{n+1} = A + \frac{y_n}{y_{n-k}}, \quad n = 0, 1, \dots, \tag{1.3}$$

where $y_{-k}, y_{-k+1}, \ldots, y_0, A \in (0, \infty)$ and $k \in \{2, 3, 4, \ldots\}$.

H.M. El-Owaidy, A.M. Ahmed, M.S. Mousa (2004) investigated the asymptotic behaviour of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \ldots, \tag{1.4}$$

where $\alpha \in [1, \infty)$, $k \in \{1, 2, \ldots\}$ and the initial conditions $x_{-k}, \ldots, x_0$ are arbitrary positive real numbers.

The equation (1.4) has also been studied by W.-S. He, W.-T. Li, X.-X. Yan (2004) but they investigated the global attractivity of (1.4) for $\alpha < 1$.

M. Aloqeili (2007) considered the difference equation

$$x_{n+1} = \alpha + \frac{x_n^p}{x_{n-1}^p}. \tag{1.5}$$

The dynamics of (1.5) with $p \in (0, 1)$ and $[0, \infty)$ were studied firstly by Aloqeili, then results of this study was generalized to the $(k + 1)th$ order difference equation

$$x_{n+1} = \alpha + \frac{x_n^p}{x_{n-k}^p}, \quad k = 2, 3, \ldots$$

CHAPTER 2

BASIC CONCEPTS AND PRELIMINARY RESULTS

In this chapter, we present some definitions and state some known results which will be useful in the subsequent chapters. The definitions given in this chapter can be found in many books [2, 5, 7, 11, 16, 18, 23, 24, 34, 35] and papers ([4, 19] and the references therein), and the preliminary results are either given in these references or can be found derived as a simple consequence of the results obtained there.

2.1 Sequences

In real analysis, a *sequence* is a function from the set of natural numbers into the set of real numbers. In other words, a sequence is a map $f : \mathbb{N} \to \mathbb{R}$. If x_n is the real number $f(n)$, instead of $f : \mathbb{N} \to \mathbb{R}$, it is usually denoted by x_n or $\{x_n\}_{n=1}^{\infty}$ or $\{x_n\}_1^{\infty}$.

Definition 2.1.1 *A sequence $\{x_n\}$ is called*

- *bounded from below if there exists $K \in \mathbb{R}$, called a lower bound, such that $x_n \geqslant K$ for all $n \in \mathbb{N}$.*
- *bounded from above if there exists $K \in \mathbb{R}$, called an upper bound, such that $x_n \leqslant K$ for all $n \in \mathbb{N}$.*
- *bounded if it is bounded both from above and from below, that is, there exists $K \in \mathbb{R}$ such that $|x_n| \leqslant K$ for all $n \in \mathbb{N}$.*
- *unbounded if it is not bounded. In other words, for all $K \in \mathbb{R}$ there exists $m \in \mathbb{N}$ such that $|x_m| > K$.*

Definition 2.1.2 *The least upper bound or supremum of a sequence x_n, which is bounded from above, is the unique real number S which satisfies both of the conditions below:*

(i) *for each $n \in \mathbb{N}$ we have $x_n \leqslant S$;*

(ii) *if $y < S$, then there exists $n \in \mathbb{N}$ such that $x_n > y$.*

Similarly, the greatest lower bound or infimum of a sequence x_n, which is bounded from below, is the unique real number I which satisfies both of conditions below:

(i) *$x_n \geqslant I$ for all $n \in \mathbb{N}$;*

(ii) *if $y > I$, then there exists $n \in \mathbb{N}$ such that $x_n < y$.*

Definition 2.1.3 *A sequence x_n is called:*

- *constant, if $x_{n+1} = x_n$ for $n = 1, 2, \ldots$;*
- *non-decreasing(increasing), if $x_{n+1} \geqslant x_n$ for $n = 1, 2, \ldots$;*
- *non-increasing(decreasing), if $x_{n+1} \leqslant x_n$ for $n = 1, 2, \ldots$;*
- *strictly increasing, if $x_{n+1} > x_n$ for $n = 1, 2, \ldots$;*
- *strictly decreasing, if $x_{n+1} < x_n$ for $n = 1, 2, \ldots$*

Definition 2.1.4 *A sequence x_n is called monotone if it is either non-increasing or non-decreasing. A sequence x_n is called strictly monotone if it is either strictly increasing or strictly decreasing.*

Definition 2.1.5 *A subsequence of a sequence is a sequence obtained by deleting some elements of the sequence preserving the relative positions of the remaining elements. More formally, if $\{x_n\}$ is a sequence and $0 < n_1 < n_2 < \cdots$ are positive integers, then the sequence $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.*

Definition 2.1.6 *A number $L \in \mathbb{R}$ is called the limit of a sequence x_n, written*

$$\lim_{n\to\infty} x_n = L,$$

if for each $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|x_n - L| < \epsilon$ for all $n \geqslant N$.

The statement $\{x_n\}$ converges to L means that the limit of x_n is a number L, and is denoted as $\{x_n\} \to L$ as $n \to \infty$. When a sequence has a limit, it is called convergent. Otherwise, it is divergent.

Lemma 2.1.7 *If a sequence is convergent, then it is bounded.*

Proof. Suppose that x_n is convergent and let $L = \lim_{n\to\infty} x_n$. Then, for $\epsilon = 1$, there is some $N \in \mathbb{N}$ such that $|x_n - L| < 1$ for all $n \geqslant N$. Therefore, $|x_n| - |L| < |x_n - L| < 1$, and hence $|x_n| < |L| + 1$. Let

$$K = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |L| + 1\}.$$

It then follows that $|x_n| \leqslant K$ for all $n \in \mathbb{N}$. Therefore x_n is bounded. □

Theorem 2.1.8 *An increasing sequence is convergent if and only if it is bounded from above. A decreasing sequence is convergent if and only if it is bounded from below.*

Proof. Let x_n be an increasing sequence which is bounded from above and let

$$s = \sup\{x_n : n \in \mathbb{N}\}.$$

Because s is the least upper bound of $\{x_n : n \in \mathbb{N}\}$, for any $\epsilon > 0$, the number $s - \epsilon$ is not an upper bound, and so there is a number x_N in the sequence for which

$$s - \epsilon < x_N.$$

Since x_n is increasing, we know that $x_N \leqslant x_n$ for all $n \geqslant N$. With s being an upper bound we can obtain

$$s - \epsilon < x_N \leqslant x_n \leqslant s < s + \epsilon \quad \text{for all} \quad n \geqslant N.$$

This is of course nothing more than $|x_n - s| < \epsilon$ for all $n \geqslant N$, which means that $\{x_n\}$ converges to s. The converse statement has been proved in Lemma 2.1.7. The second assersion can be proved in a similar way. So, the proof of that part is omitted here. □

Definition 2.1.9 *Let x_n be any sequence of real numbers. The limit supremum of this sequence is the greatest limit of all subsequences of the given sequence. More rigorously, for each n let*

$$a_n = \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\} \tag{2.1}$$

Then a_n is a monotone decreasing sequence (since as n becomes large we are taking the supremum of smaller set of numbers), so it has a limit.

Limit supremum of x_n is defined to be

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} a_n$$

The limit supremum may be $\pm\infty$.

Likewise, the limit infimum of the given sequence is the least limit of all subsequences of the given sequence. In detail, let

$$b_n = \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\} \tag{2.2}$$

Then b_n is a monotone increasing sequence (since as n becomes large we are taking the infimum of a smaller set of numbers), so it has a limit.We define the limit infimum of x_n to be

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} b_n,$$

which also may be $\pm\infty$.

Proposition 1 *If x_n is a bounded sequence, then $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$ both exist, and further*

$$\liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n.$$

Proof. By (2.1) and (2.2), we have $b_n \leqslant x_n \leqslant a_n$ and since x_n is bounded there exist constants K and M such that $K \leqslant x_n \leqslant M$ for all n. Now, $b_n \leqslant x_n \leqslant M$ means that the sequence b_n is bounded from above, and as b_n is increasing, by Theorem 2.1.8, it converges. Similarly, a_n converges. That is, $\lim b_n$ and $\lim a_n$ exist and, clearly, $\lim b_n \leqslant \lim a_n$, which is the required result. □

Remark 2.1.10

1. *$\{x_n\}$ is bounded if and only if its upper and lower limits are finite:*

$$-\infty < \liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n < +\infty.$$

2. *For all $\epsilon > 0$ all terms of a bounded sequence with the exception of a finitely many of them are located in the interval:*

$$(\liminf_{n\to\infty} x_n - \epsilon, \limsup_{n\to\infty} x_n + \epsilon),$$

or, equivalently, all the points (n, x_n) with the exception of a finitely many of them are located between the lines

$$y = \liminf_{n\to\infty} x_n - \epsilon \quad \text{and} \quad y = \limsup_{n\to\infty} x_n + \epsilon.$$

3. *For all $\epsilon > 0$, infinitely many terms of x_n are above the line $y = \liminf_{n\to\infty} x_n - \epsilon$ and below the line $y = \limsup_{n\to\infty} x_n + \epsilon$.*

2.2 Difference Equations

Definition 2.2.1 *A difference equation is an equation of the form*

$$x_{n+1} = f(x_n), \quad n = 0, 1, \ldots \tag{2.3}$$

where f is a continuous function which maps some set J into J. The set J is usually an interval of real numbers, or a union of intervals, it may even be a discrete set.

If the function f in (2.3) *is replaced by a function $g : \mathbb{N} \times J \to J$, then we have*

$$x_{n+1} = g(n, x_n), \quad n = 0, 1, \ldots \tag{2.4}$$

Equation (2.4) *is called nonautonomous, while equation* (2.3) *is called autonomous.*

The right sides of (2.3) and (2.4) depend only on x_n. That is, to find the value at any stage, we need only the previous value. However, if the value at any stage depends not only on the previous stage but some earlier stages as well, then we have an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots \tag{2.5}$$

in which $k \geqslant 0$ is an integer. (2.5) is a difference equation of order $k+1$.

A solution of (2.5) is a sequence $\{x_n\}_{n=-k}^{\infty}$ which satisfies (2.5) for all $n \geqslant 0$. If we prescribe a set of $k+1$ initial conditions

$$x_{-k}, x_{-k+1}, \dots, x_0 \in J,$$

then

$$x_1 = f(x_0, x_{-1}, \dots, x_{-k})$$
$$x_2 = f(x_1, x_0, \dots, x_{-k+1})$$
$$\vdots$$

and so the solution $\{x_n\}_{n=-k}^{\infty}$ of (2.5) exists for all $n \geqslant -k$ and is uniquely determined by the initial conditions.

Definition 2.2.2 *A solution of* (2.5) *which is a constant for all* $n \geqslant -k$ *is called equilibrium solution of* (2.5). *If*

$$x_n = \bar{x} \quad \text{for all} \quad n \geqslant -k$$

is a constant solution of (2.5), *then* $\bar{x}$ *is called an equilibrium point, or a fixed point of* (2.5).

Definition 2.2.3 *Let* $\{x_n\}_{n=-k}^{\infty}$ *be a solution of* (2.5).

- *A positive semi-cycle of* $\{x_n\}_{n=-1}^{\infty}$ *consists of a string of terms* $\{x_l, x_{l+1}, ..., x_m\}$, *all greater than or equal to* $\bar{x}$, *with* $l \geqslant -k$ *and* $m \leqslant \infty$ *and such that*

$$\text{either } l = -k, \quad \text{or} \quad l > -k \text{ and } x_{l-1} < \bar{x}$$

and

$$\text{either } m = \infty, \quad \text{or} \quad m < \infty \text{ and } x_{m+1} < \bar{x}.$$

- *A negative semi-cycle of* $\{x_n\}_{n=-1}^{\infty}$ *consists of a string of terms* $\{x_l, x_{l+1}, ..., x_m\}$, *all less than* $\bar{x}$, *with* $l \geqslant -k$ *and* $m \leqslant \infty$ *and such that*

$$\text{either } l = -k, \quad \text{or} \quad l > -k \text{ and } x_{l-1} \geqslant \bar{x}$$

and

$$\text{either } m = \infty, \quad \text{or} \quad m < \infty \text{ and } x_{m+1} \geqslant \bar{x}.$$

Definition 2.2.4 *A solution* $\{x_n\}_{n=-k}^{\infty}$ *of (2.5) is called non-oscillatory if there exists* $N \geqslant -k$ *such that either*

$$x_n > \bar{x} \quad \text{for all} \quad n \geqslant N$$

or

$$x_n < \bar{x} \quad \text{for all} \quad n \geqslant N.$$

A solution $\{x_n\}_{n=-k}^{\infty}$ *is called oscillatory if it is not non-oscillatory.*

Definition 2.2.5 *A solution* $\{x_n\}_{n=-k}^{\infty}$ *of* (2.5) *is said to be periodic with period p (or a period p solution) if there exists an integer* $p \geqslant 1$ *such that*

$$x_{n+p} = x_n \quad \text{for all} \quad n \geqslant -k. \tag{2.6}$$

The solution is said to be periodic with prime p if p is the smallest positive integer for which equation (2.6) *holds. In this case, a* p*-tuple*

$$(x_{n+1}, x_{n+2}, \ldots, x_{n+p})$$

of any p *consecutive values of the solution is called a* p*-cycle of equation* (2.5)*. A solution* $\{x_n\}_{n=-k}^{\infty}$ *of* (2.5) *is called eventually periodic with period* p *if there exists an integer* $N \geqslant -k$ *such that* $\{x_n\}_{n=N}^{\infty}$ *is periodic with period* p*; that is,*

$$x_{n+p} = x_n \text{ for all } n \geqslant N.$$

Definition 2.2.6 *A fixed point* $\bar{x}$ *of* (2.5) *is said to be:*

- *locally stable if there exists an interval* $I \subset (0, \infty)$ *such that to any* $\epsilon > 0$ *there corresponds a* $\delta = \delta(\epsilon) > 0$ *with the property that*

  $$x_{-k}, \ldots, x_0 \in I \text{ and } |x_{-k} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \delta \quad \text{implies} \quad |x_n - \bar{x}| < \epsilon$$

  for all $n \geqslant -k$*;*

- *locally asymptotically stable if it is locally stable and there exists* $\gamma > 0$ *such that*

  $$x_{-k}, \ldots, x_0 \in I \text{ and } |x_{-k} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \gamma \quad \text{implies} \quad \lim_{n\to\infty} x_n = \bar{x};$$

- *global attractor if* $\lim_{n\to\infty} x_n = \overline{x}$ *for any* $x_{-k}, \ldots, x_0 \in I$;

- *globally asymptotically stable if it is locally asymptotically stable and a global attractor;*

- *unstable if it is not locally stable.*

- *source if there exists* $r > 0$ *such that for every solution of equation* (2.5) *with*

$$0 < |x_{-k} - \overline{x}| + \cdots + |x_0 - \overline{x}| < r,$$

there exists $N \geqslant 1$ *such that* $|x_N - \overline{x}| > r$. *Clearly a source is an unstable equilibrium point of equation* (2.5).

In general, the local stability analysis of the fixed point of a nonlinear equation is carried out by means of linearization about the fixed point.

Suppose the function f in (2.5) is continuously differentiable in some open neighborhood of $(\overline{x}, \ldots, \overline{x})$. Let

$$p_i = \frac{\partial f}{\partial u_i}(\overline{x}, \overline{x}, \ldots, \overline{x}) \quad \text{for} \quad i = 0, 1, \ldots, k$$

denote the partial derivative of $f(u_0, u_1, \ldots, u_k)$ with respect to u_i. Then the equation

$$y_{n+1} = p_0 y_n + p_1 y_{n-1} + \cdots + p_k y_{n-k}, \quad n = 0, 1, \ldots \tag{2.7}$$

is called the "linearized equation" of (2.5) about the equilibrium point $\overline{x}$, and the equation

$$\lambda^{k+1} - p_0\lambda^k - \cdots - p_{k-1}\lambda - p_k = 0 \tag{2.8}$$

is called the "characteristic equation" of (2.7).

The following well-known result, called the Linearized Stability Theorem, is very useful in performing the local stability analysis of the equilibrium point $\overline{x}$ of (2.5).

Theorem 2.2.7 *(The Linearized Stability Theorem) Suppose f is a continuously differentiable function defined on some open neighborhood of* $\overline{x}$. *Then the following statements are true:*

1. *If all the roots of equation* (2.8) *have absolute value less than one, then the equilibrium point* $\bar{x}$ *of* (2.7) *is locally asymptotically stable.*

2. *If at least one root of equation* (2.8) *has absolute value greater than one, then the equilibrium point* $\bar{x}$ *of* (2.7) *is unstable.*

3. *If all the roots of equation* (2.8) *have absolute value greater than one, then the equilibrium point* $\bar{x}$ *of* (2.7) *is a source.*

Proof. For a detailed proof of this theorem see [11]. □

The equilibrium point $\bar{x}$ of equation (2.5) is called "hyperbolic" if no root of equation (2.8) has absolute value equal to one. If there exists a root of equation (2.8) with absolute value equal to one, then $\bar{x}$ is called "non-hyperbolic".

The equilibrium point $\bar{x}$ of equation (2.5) is called a "sink" if every root of (2.8) has absolute value less than one. Thus a sink is locally asymptotically stable, but converse need not be true.

The equilibrium point $\bar{x}$ of equation (2.5) is called a "saddle point" equilibrium point if it is hyperbolic, and if in addition, there exists a root of (2.8) with absolute value less than one and another root of (2.8) with absolute value greater than one. In particular, a saddle point equilibrium point is unstable.

2.2.1 Second Order Autonomous Difference Equations

A second order autonomous difference equation can be written as

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots. \tag{2.9}$$

Let $\bar{x}$ be an equilibrium point of (2.9). Suppose that the function $f(u_0, u_1)$ in (2.9) is continuously differentiable in some neighborhood of $(\bar{x}, \bar{x})$. Setting $A = \frac{\partial f}{\partial u_0}(\bar{x}, \bar{x})$ and $B = \frac{\partial f}{\partial u_1}(\bar{x}, \bar{x})$, one can write the so-called linearized equation of (2.9) about $\bar{x}$ as

$$y_{n+1} = Ay_n + By_{n-1}, \quad n = 0, 1, \ldots, \tag{2.10}$$

whose characteristic equation is

$$\lambda^2 - A\lambda - B = 0. \tag{2.11}$$

Let λ_1 and λ_2 denote the roots of (2.11). Then, the following *Linearized Stability Theorem* holds [24, Theorem 1.1.1]:

Theorem 2.2.8 *(Linearized Stability)*

(i) If $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$, *then the fixed point* $\bar{x}$ *of* (2.9) *is locally asymptotically stable.*

(ii) If $|\lambda_1| > 1$ *or* $|\lambda_2| > 1$, *then the fixed point* $\bar{x}$ *of* (2.9) *is unstable.*

(iii) A necessary and sufficient condition for both roots of (2.11) *to lie in the open unit disk* $|\lambda| < 1$ *is*

$$|A| < 1 - B < 2.$$

(iv) A necessary and sufficient condition for at least one root of (2.11) *to lie out of the open unit disk is*

$$|B| > 1 \quad \text{and} \quad |A| < |1 - B|$$

or

$$A^2 + 4B > 0 \quad \text{and} \quad |A| > |1 - B|.$$

Proof. (i) and (ii) are special cases of Theorem 2.2.7 (i) and (ii). The other parts can be proven by analyzing the roots of (2.11). □

The following theorem, also given in [24], will be useful to obtain the global asymptotic stability condition of the fixed point $\bar{x}$ of (2.9).

Theorem 2.2.9 *[24] Let* $f : (0,\infty) \times (0,\infty) \to (0,\infty)$ *be a continuous function and consider the difference equation* (2.9) *Suppose* f *satisfies the following conditions:*

- *There exist positive numbers a and b with* $a < b$ *such that*

$$a \leqslant f(x,y) \leqslant b \quad \text{for all} \quad x, y \in [a,b]\,;$$

- $f(x,y)$ *is non-increasing in* $x \in [a,b]$ *for each* $y \in [a,b]$, *and* $f(x,y)$ *is non-decreasing in* $y \in [a,b]$ *for each* $x \in [a,b]$*;*

- *Equation* (2.9) *has no prime 2-periodic solutions in [a,b].*

Then, there exists exactly one equilibrium point $\overline{x}$ *of* (2.9) *which lies in [a,b]. Moreover, every solution of* (2.9) *which lies in* $[a, b]$ *converges to* $\overline{x}$*.*

CHAPTER 3

MAIN RESULTS

In this chapter, we investigate the boundedness character and the semi-cycle analysis of the positive solutions, the periodic nature and the stability of a second order nonlinear difference equation. Some examples illustrating the obtained results is constructed as well.

3.1 Introduction

The aim in this chapter is to examine the boundedness and semi-cycle analysis of positive solutions, existence of prime 2-periodic solutions, local and global asymptotic stability of the recursive sequence

$$x_{n+1} = \alpha + \beta x_{n-1} + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \ldots \tag{3.1}$$

where $\alpha \in [0, \infty)$, $\beta \in [0, 1)$ and initial conditions x_{-1} and x_0 are arbitrary positive real numbers. Equation(3.1) with the special case $\beta = 0$ becomes

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \ldots \tag{3.2}$$

which has been dealt with by many authors. The recursive sequence (3.2) for negative values of α has been examined in [19, 32], and that for non-negative values of α has been studied in [4]. This thesis is concerned with (3.1) which is more general than (3.2). The results presented here are true for (3.2) as well. Some of the current results, upon setting $\beta = 0$, are already known for (3.2).

The next section contains some lemmas and a corollary which are necessary for boundedness and semi-cycle analysis. Third section is about the boundedness of positive solutions of (3.1). In section four, the prime 2-periodic solutions of (3.1) are studied and the semi-cycle analysis of positive solutions is considered as well. Next section is related to the stability conditions of the fixed point of (3.1). This chapter concludes with some numeric examples to illustrate the theoretical results in the last section.

3.2 Auxiliary Results

Before starting the main results, some auxiliary results are presented here. These outcomes will pave the way in achieving the final conclusions.

Lemma 3.2.1 *Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of (3.1). Then, the following statements are true for all n.*

(i) $x_{n+1} > x_{n-1} \Leftrightarrow x_{n-1} + \alpha x_n + (\beta - 1)x_{n-1}x_n > 0.$

(ii) $x_{n+1} = x_{n-1} \Leftrightarrow x_{n-1} + \alpha x_n + (\beta - 1)x_{n-1}x_n = 0.$

(iii) $x_{n+1} < x_{n-1} \Leftrightarrow x_{n-1} + \alpha x_n + (\beta - 1)x_{n-1}x_n < 0.$

Proof. The conclusions follow immediately from the fact that

$$\begin{aligned} x_{n+1} - x_{n-1} &= \alpha + \beta x_{n-1} + \frac{x_{n-1}}{x_n} - x_{n-1} \\ &= \frac{\alpha x_n + \beta x_{n-1}x_n + x_{n+1} - x_{n-1}x_n}{x_n} \\ &= \frac{x_{n-1} + \alpha x_n + (\beta - 1)x_{n-1}x_n}{x_n}. \end{aligned}$$

(i) If $x_{n+1} - x_{n-1} > 0$, then $x_{n+1} < x_{n-1} \Leftrightarrow x_{n-1} + \alpha x_n + (\beta - 1)x_{n-1}x_n > 0$ since $\{x_n\}_{n=-1}^{\infty}$.

(ii) If $x_{n+1} - x_{n-1} = 0$, then $x_{n+1} = x_{n-1} \Leftrightarrow x_{n-1} + \alpha x_n + (\beta - 1)x_{n-1}x_n = 0$ since $\{x_n\}_{n=-1}^{\infty}$.

(iii) If $x_{n+1} - x_{n-1} < 0$, then $x_{n+1} > x_{n-1} \Leftrightarrow x_{n-1} + \alpha x_n + (\beta - 1)x_{n-1}x_n < 0$ since $\{x_n\}_{n=-1}^{\infty}$.

$\square$

Corollary 3.2.2 *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution* (3.1), *and* $\alpha = 1$. *Then*

(i) If $x_{-1} < x_1$, *then* $x_{-1} < x_1 < x_3 < \cdots$ *and* $x_0 < x_2 < x_4 < \cdots$;

(ii) If $x_{-1} = x_1$, *then* $x_{-1} = x_1 = x_3 = \cdots$ *and* $x_0 = x_2 = x_4 = \cdots$;

(iii) If $x_{-1} > x_1$, *then* $x_{-1} > x_1 > x_3 > \cdots$ *and* $x_0 > x_2 > x_4 > \cdots$.

Proof. Observe that, for $n \geqslant 0$,

$$\begin{aligned}
x_n + x_{n+1} + (\beta-1)x_nx_{n+1} &= x_n + 1 + \beta x_{n-1} + \frac{x_{n-1}}{x_n} + (\beta-1)x_n\left(1 + \beta x_{n-1} + \frac{x_{n-1}}{x_n}\right) \\
&= x_n + 1 + \beta x_{n-1} + \frac{x_{n-1}}{x_n} + \beta x_n + \beta^2 x_{n-1}x_n + \beta x_{n-1} - x_n - \beta x_{n-1}x_n - x_{n-1} \\
&= \frac{x_n + \beta x_{n-1}x_n + x_{n-1} + \beta x_n^2 + \beta^2 x_{n-1}x_n^2 + \beta x_{n-1}x_n - \beta x_{n-1}x_n^2 - x_{n-1}x_n}{x_n} \\
&= \frac{x_{n-1} + x_n + (\beta-1)x_{n-1}x_n + \beta x_n(x_{n-1} + x_n + (\beta-1)x_{n-1}x_n)}{x_n} \\
&= \frac{(1+\beta x_n)(x_{n-1} + x_n + (\beta-1)x_{n-1}x_n)}{x_n} \qquad (3.3)
\end{aligned}$$

and consider Lemma 3.2.1 with $\alpha = 1$. Now,

(i) if $x_{-1} < x_1$, then by Lemma 3.2.1 (i) for $n = 0$,

$$x_{-1} + x_1 + (\beta-1)x_{-1}x_0 > 0$$

Hence by (3.3) with $n = 0$

$$x_0 + x_1 + (\beta-1)x_0x_1 = \frac{(1+\beta x_0)(x_{-1} + x_0 + (\beta-1)x_{-1}x_0)}{x_0} > 0.$$

Therefore, taking $n = 1$ in Lemma 3.2.1 (i), $x_2 > x_0$

Now, assume that $x_{k-1} < x_{k+1}$. Then by Lemma 3.2.1 (i) with $n = k$

$$x_{k-1} + x_{k+1} + (\beta-1)x_{k-1}x_{k+1} > 0.$$

Then by equation (3.3) with $n = k$

$$x_k + x_{k+1} + (\beta-1)x_kx_{k+1} = \frac{(1+\beta x_k)(x_{k-1} + x_k + (\beta-1)x_{k-1}x_k)}{x_{k+3}} > 0.$$

Therefore, by Lemma 3.2.1 (i) $x_k < x_{k+2}$ with $n = k + 1$.

The other parts can be proved in a similar way. □

Theorem 2 of [33] states that if α_n is a two-periodic sequence, f and g are non-decreasing continuous functions which map the interval $(0, \infty)$ into itself, and $\{x_n\}$ is a positive solution of

$$x_n = \alpha_n + \frac{f(x_{n-2})}{g(x_{n-1})}, \tag{3.4}$$

then the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are eventually monotone.

Taking $\alpha_n = \alpha$, $f(x) = x$ and $g(x) = x/(\beta x + 1)$, in Theorem 2 of [33], the following result can be deduced.

Lemma 3.2.3 *Let* $\alpha \geqslant 0$, $0 \leqslant \beta < 1$, *and* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of* (3.1). *Then,* $\{x_{2n}\}_{n=0}^{\infty}$ *and* $\{x_{2n-1}\}_{n=0}^{\infty}$ *are eventually monotone.*

Proof. From (3.1), one can write

$$x_{2n+1} - x_{2n-1} = \beta(x_{2n-1} - x_{2n-3}) + \frac{x_{2n-1}}{x_{2n}} - \frac{x_{2n-3}}{x_{2n-2}}, \quad n \geqslant 1, \tag{3.5}$$

and

$$x_{2n+2} - x_{2n} = \beta(x_{2n} - x_{2n-2}) + \frac{x_{2n}}{x_{2n+1}} - \frac{x_{2n-2}}{x_{2n-1}}, \quad n \geqslant 1. \tag{3.6}$$

Now, there are four cases to consider:

Case 1: If $x_{-1} \leqslant x_1$ and $x_0 \geqslant x_2$, then, from (3.5), one gets

$$x_3 - x_1 = \beta(x_1 - x_{-1}) + \frac{x_1}{x_2} - \frac{x_{-1}}{x_0} \geqslant \frac{x_1}{x_2} - \frac{x_{-1}}{x_0} \geqslant \frac{x_1 - x_{-1}}{x_0} \geqslant 0,$$

that is, $x_1 \leqslant x_3$. Moreover, from (3.6), one gets

$$x_4 - x_2 = \beta(x_2 - x_0) + \frac{x_2}{x_3} - \frac{x_0}{x_1} \leqslant \frac{x_2}{x_3} - \frac{x_0}{x_1} \leqslant \frac{x_2 - x_0}{x_1} \leqslant 0,$$

that is, $x_2 \geqslant x_4$. By induction, one sees that

$$x_{-1} \leqslant x_1 \leqslant x_3 \leqslant \cdots \leqslant x_{2n+1} \leqslant \cdots \quad \text{and} \quad x_0 \geqslant x_2 \geqslant x_4 \geqslant \cdots \geqslant x_{2n} \geqslant \cdots.$$

Case 2: The case $x_{-1} \geqslant x_1$ and $x_0 \leqslant x_2$, then from (3.5)

$$x_3 - x_1 = \beta(x_1 - x_{-1}) + \frac{x_1}{x_2} - \frac{x_{-1}}{x_0} \leqslant \frac{x_1}{x_2} - \frac{x_{-1}}{x_0} \leqslant \frac{x_1 - x_{-1}}{x_2} \leqslant 0,$$

that is $x_3 \leqslant x_1$. Moreover from (3.6)

$$x_4 - x_2 = \beta(x_2 - x_0) + \frac{x_2}{x_3} - \frac{x_0}{x_1} \geqslant \frac{x_2}{x_3} - \frac{x_0}{x_1} \geqslant \frac{x_2 - x_0}{x_1} \geqslant 0,$$

that is $x_4 \geqslant x_2$. By induction, one sees that

$$x_{-1} \geqslant x_1 \geqslant x_3 \geqslant \cdots \geqslant x_{2n+1} \geqslant \cdots \quad \text{and} \quad x_0 \leqslant x_2 \leqslant x_4 \leqslant \cdots \leqslant x_{2n} \leqslant \cdots .$$

Case 3: Assume that $x_{-1} \leqslant x_1$ and $x_0 \leqslant x_2$. If $x_1 \geqslant x_3$, then similar to Case 1 we can obtain

$$x_0 \leqslant x_2 \leqslant x_4 \leqslant \cdots \leqslant x_{2n} \leqslant \cdots \quad \text{and} \quad x_1 \geqslant x_3 \geqslant x_5 \geqslant \cdots \geqslant x_{2n+1} \geqslant \cdots .$$

That is why, we may assume that $x_{-1} \leqslant x_1 \leqslant x_3$ and $x_0 \leqslant x_2$. If $x_2 \geqslant x_4$, then similarly we can obtain

$$x_1 \leqslant x_3 \leqslant x_5 \leqslant \cdots \leqslant x_{2n+1} \leqslant \cdots \quad \text{and} \quad x_2 \geqslant x_4 \geqslant x_6 \geqslant \cdots \geqslant x_{2n} \geqslant \cdots .$$

So we may assume that $x_{-1} \leqslant x_1 \leqslant x_3$ and $x_0 \leqslant x_2 \leqslant x_4$.

Continuing in this way we have that, there is $k \in \mathbb{N}$ such that

$$x_{-1} \leqslant x_1 \leqslant x_3 \leqslant \cdots \leqslant x_{2k-1}, \quad x_0 \leqslant x_2 \leqslant x_4 \leqslant \cdots \leqslant x_{2k}, \quad x_{2k+1} \leqslant x_{2k-1} \tag{3.7}$$

or

$$x_{-1} \leqslant x_1 \leqslant x_3 \leqslant \cdots \leqslant x_{2k+1}, \quad x_0 \leqslant x_2 \leqslant x_4 \leqslant \cdots \leqslant x_{2k}, \quad x_{2k+2} \leqslant x_{2k} \tag{3.8}$$

or there is no such k, that is,

$$x_{-1} \leqslant x_1 \leqslant x_3 \leqslant \cdots \leqslant x_{2n+1} \leqslant \cdots , \quad x_0 \leqslant x_2 \leqslant x_4 \leqslant \cdots \leqslant x_{2n} \leqslant \cdots , \tag{3.9}$$

which means that $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are monotone.

If (3.7) holds, then similar to Case 2 we have

$$x_0 \leqslant x_2 \leqslant x_4 \leqslant \cdots \leqslant x_{2n} \leqslant \cdots \quad \text{and} \quad x_{2k-1} \geqslant x_{2k+1} \geqslant \cdots \geqslant x_{2(n+k)-1} \geqslant \cdots .$$

On the other hand, if (3.8) holds, then similar to Case 1 we have

$$x_{-1} \leqslant x_1 \leqslant x_3 \leqslant \cdots \leqslant x_{2n+1} \leqslant \cdots \quad \text{and} \quad x_{2k} \geqslant x_{2k+2} \geqslant \cdots \geqslant x_{2(n+k)} \geqslant \cdots .$$

This gives us the required result in this case.

Case 4: Assume that $x_{-1} \geqslant x_1$ and $x_0 \geqslant x_2$. If $x_1 \leqslant x_3$, then similar to Case 2 we can obtain

$$x_0 \geqslant x_2 \geqslant x_4 \geqslant \cdots \geqslant x_{2n} \geqslant \cdots \quad \text{and} \quad x_1 \leqslant x_3 \leqslant x_5 \leqslant \cdots \leqslant x_{2n+1} \leqslant \cdots .$$

That is why, we may assume that $x_{-1} \geqslant x_1 \geqslant x_3$ and $x_0 \geqslant x_2$. If $x_2 \leqslant x_4$, then similarly we can obtain

$$x_1 \geqslant x_3 \geqslant x_5 \geqslant \cdots \geqslant x_{2n+1} \geqslant \cdots \quad \text{and} \quad x_2 \leqslant x_4 \leqslant x_6 \leqslant \cdots \leqslant x_{2n} \leqslant \cdots .$$

So we may assume that $x_{-1} \geqslant x_1 \geqslant x_3$ and $x_0 \geqslant x_2 \geqslant x_4$.

Continuing in this way we have that, there is $k \in \mathbb{N}$ such that

$$x_{-1} \geqslant x_1 \geqslant x_3 \geqslant \cdots \geqslant x_{2k-1}, \quad x_0 \geqslant x_2 \geqslant x_4 \geqslant \cdots \geqslant x_{2k}, \quad x_{2k+1} \geqslant x_{2k-1} \tag{3.10}$$

or

$$x_{-1} \geqslant x_1 \geqslant x_3 \geqslant \cdots \geqslant x_{2k+1}, \quad x_0 \geqslant x_2 \geqslant x_4 \geqslant \cdots \geqslant x_{2k}, \quad x_{2k+2} \geqslant x_{2k} \tag{3.11}$$

or there is no such k, that is,

$$x_{-1} \geqslant x_1 \geqslant x_3 \geqslant \cdots \geqslant x_{2n+1} \geqslant \cdots, \quad x_0 \geqslant x_2 \geqslant x_4 \geqslant \cdots \geqslant x_{2n} \geqslant \cdots, \tag{3.12}$$

which means that $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are monotone.

If (3.10) holds, then similar to Case 1 we have

$$x_0 \geqslant x_2 \geqslant x_4 \geqslant \cdots \geqslant x_{2n} \geqslant \cdots \quad \text{and} \quad x_{2k-1} \leqslant x_{2k+1} \leqslant \cdots \leqslant x_{2(n+k)-1} \leqslant \cdots .$$

On the other hand, if (3.11) holds, then similar to Case 2 we have

$$x_{-1} \geqslant x_1 \geqslant x_3 \geqslant \cdots \geqslant x_{2n+1} \geqslant \cdots \quad \text{and} \quad x_{2k} \leqslant x_{2k+2} \leqslant \cdots \leqslant x_{2(n+k)} \leqslant \cdots .$$

This gives us the required result in this case. □

Lemma 3.2.4 *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of* (3.1), *and let $L > \frac{\alpha}{1-\beta}$. Then,*

(i) $\lim_{n\to\infty} x_{2n} = L$ *if and only if* $\lim_{n\to\infty} x_{2n+1} = \dfrac{L}{(1-\beta)L-\alpha}$.

(ii) $\lim_{n\to\infty} x_{2n+1} = L$ *if and only if* $\lim_{n\to\infty} x_{2n} = \dfrac{L}{(1-\beta)L-\alpha}$.

Proof. From (3.1), one has

$$x_{2n+2} = \alpha + \beta x_{2n} + \frac{x_{2n}}{x_{2n+1}} \tag{3.13}$$

and

$$x_{2n+1} = \alpha + \beta x_{2n-1} + \frac{x_{2n-1}}{x_{2n}} \tag{3.14}$$

If $\lim_{n\to\infty} x_{2n} = L$, then taking limit of (3.13) as $n \to \infty$ one obtains,

$$L = \alpha + \beta L + \frac{L}{\lim_{n\to\infty} x_{2n+1}}$$

which gives

$$\lim_{n\to\infty} x_{2n+1} = \frac{L}{(1-\beta)L - \alpha}.$$

Conversely, assume that $\lim_{n\to\infty} x_{2n+1} = L/[(1-\beta)L - \alpha]$, then taking limit of (3.13) as $n \to \infty$ one obtains,

$$\lim_{n\to\infty} x_{2n+2} = \alpha + \beta \lim_{n\to\infty} x_{2n} + \frac{\lim_{n\to\infty} x_{2n}}{L/[(1-\beta)L - \alpha]}$$

which gives us

$$\lim_{n\to\infty} x_{2n} = L.$$

If $\lim_{n\to\infty} x_{2n+1} = L$, then taking limit of (3.14) as $n \to \infty$ one obtains,

$$L = \alpha + \beta L + \frac{L}{\lim_{n\to\infty} x_{2n}}$$

which gives

$$\lim_{n\to\infty} x_{2n} = \frac{L}{(1-\beta)L - \alpha}$$

Conversely, assume that $\lim_{n\to\infty} x_{2n} = L/[(1-\beta)L - \alpha]$, then taking limit of (3.14) as $n \to \infty$ one obtains,

$$\lim_{n\to\infty} x_{2n+1} = \alpha + \beta \lim_{n\to\infty} x_{2n-1} + \frac{\lim_{n\to\infty} x_{2n-1}}{L/[(1-\beta)L - \alpha]}$$

which gives us

$$\lim_{n\to\infty} x_{2n+1} = L.$$

□

3.3 Boundedness

In this part, the boundedness of positive solutions of (3.1) is addressed. For this purpose, firstly the following lemma which will be important to prove the existence of an unbounded solution has been provided.

Lemma 3.3.1 *Let* $\alpha \geqslant 0$, $0 \leqslant \beta < 1$, *and* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of* (3.1). *Then, at least one of the subsequences* $\{x_{2n}\}_{n=0}^{\infty}$ *and* $\{x_{2n-1}\}_{n=0}^{\infty}$ *is bounded. Moreover,*

(i) $\lim_{n\to\infty} x_{2n} = \infty$ *if and only if* $\lim_{n\to\infty} x_{2n-1} = \dfrac{\alpha}{1-\beta}$.

(ii) $\lim_{n\to\infty} x_{2n-1} = \infty$ *if and only if* $\lim_{n\to\infty} x_{2n} = \dfrac{\alpha}{1-\beta}$.

Proof. Suppose that $\{x_n\}_{n=-1}^{\infty}$ is a positive solution of (3.1) such that both $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ are unbounded. Using Lemma 3.2.3, it is easy to see that $\lim_{n\to\infty} x_{2n} = \infty$ and $\lim_{n\to\infty} x_{2n-1} = \infty$. Then,

$$\lim_{n\to\infty} \frac{x_{2n+1}}{x_{2n-1}} = \lim_{n\to\infty} \left(\frac{\alpha}{x_{2n-1}} + \beta + \frac{1}{x_{2n}} \right) = \beta.$$

Now, for $\epsilon = (1-\beta)/2$, there exists $N \in \mathbb{N}$ such that

$$\left| \frac{x_{2n+1}}{x_{2n-1}} - \beta \right| < \frac{1-\beta}{2} \quad \text{for all} \quad n > N,$$

which gives us $x_{2n+1} < \frac{1+\beta}{2} x_{2n-1}$ for all $n > N$. Using this inequality repeatedly, one obtains

$$x_{2n+1} < \left(\frac{1+\beta}{2} \right)^{n-N} x_{2N+1} \quad \text{for all} \quad n > N.$$

Since $(1+\beta)/2 < 1$, the above estimate leads to $\lim_{n\to\infty} x_{2n+1} = 0$, which is a contradiction. Additionally, one can show that if one of the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ is unbounded, then the other one converges to $\alpha/(1-\beta)$. □

In the next theorem, it is shown that there exist positive solutions of (3.1) which are unbounded.

Theorem 3.3.2 *Let* $0 \leqslant \alpha < 1$, $0 \leqslant \beta < 1$, *and* $\{x_n\}_{n=-1}^{\infty}$ *be a solution of (3.1) such that* $0 < x_{-1} < \frac{1}{1-\beta}$ *and* $x_0 > \frac{1}{(1-\alpha)(1-\beta)}$. *Then,*

$$\lim_{n\to\infty} x_{2n} = \infty \quad \text{and} \quad \lim_{n\to\infty} x_{2n+1} = \frac{\alpha}{1-\beta}. \tag{3.15}$$

Proof. Since $0 \leqslant \alpha < 1$, it is clear that $\frac{1}{1-\alpha} \geqslant \alpha + 1$, hence $x_0 > \bar{x}$. Moreover,

$$x_1 = \alpha + \beta x_{-1} + \frac{x_{-1}}{x_0} < \frac{1}{1-\beta}$$

and

$$x_1 = \alpha + \beta x_{-1} + \frac{x_{-1}}{x_0} > \alpha.$$

That is, $\alpha < x_1 < \frac{1}{1-\beta}$. On the other hand,

$$x_2 = \alpha + \beta x_0 + \frac{x_0}{x_1} = \alpha + \left(\beta + \frac{1}{x_1}\right) x_0 > \alpha + x_0,$$
$$x_3 = \alpha + \beta x_1 + \frac{x_1}{x_2} < \alpha + \beta x_1 + \frac{x_1}{x_0} < \frac{1}{1-\beta}.$$

By mathematical induction, one can show that

$$x_{2n} > n\alpha + x_0 \quad \text{and} \quad x_{2n-1} \in \left(\alpha, \frac{1}{1-\beta}\right) \quad \text{for all} \quad n \geqslant 1. \tag{3.16}$$

Therefore, if $\alpha \neq 0$, then $\lim_{n\to\infty} x_{2n} = \infty$ and, hence, by Lemma 3.3.1, one obtains $\lim_{n\to\infty} x_{2n+1} = \alpha/(1-\beta)$ as claimed.

On the other hand, for $\alpha = 0$, one has

$$x_{2n+2} - x_{2n} = \left(\beta - 1 + \frac{1}{x_{2n+1}}\right) x_{2n} > 0$$

and

$$x_{2n+1} - x_{2n-1} = \left(\beta - 1 + \frac{1}{x_{2n}}\right) x_{2n-1} < 0$$

which mean that $\{x_{2n}\}$ is strictly increasing and $\{x_{2n+1}\}$ is strictly decreasing. Therefore, if $\lim_{n\to\infty} x_{2n} = L < \infty$, then by Lemma 3.2.4 one obtains $\lim_{n\to\infty} x_{2n+1} = 1/(1-\beta)$. Taking the limit as $n \to \infty$ on both sides of $x_{2n+1} = (\beta + \frac{1}{x_{2n}})x_{2n-1}$ yields $L = 1/(1-\beta)$, which is not possible since $\{x_{2n}\}$ is increasing and $x_0 \geqslant \bar{x}$. Therefore, $\lim_{n\to\infty} x_{2n} = \infty$ and, by Lemma 3.3.1, $\lim_{n\to\infty} x_{2n+1} = 0$ as required. □

3.4 Periodicity and Semi-cycle Analysis

In this part, the case where the solutions of (3.1) are prime 2-periodic is considered. Also, the semi-cycle analysis of positive solutions is done by means of which the convergence of any positive solution to the fixed point or to a prime 2-periodic solution of (3.1) is dealt with alongside.

Lemma 3.4.1 *Equation (3.1) has prime 2-periodic solutions if and only if* $\alpha = 1$. *Moreover, when* $\alpha = 1$, $\{x_n\}_{n=-1}^{\infty}$ *is prime 2-periodic if and only if* $x_{-1} \neq \frac{2}{1-\beta}$, $x_{-1} \neq \frac{1}{1-\beta}$ *and* $x_0 = \frac{x_{-1}}{x_{-1}(1-\beta)-1}$.

Proof. Suppose that (3.1) has a 2-periodic solution

$$\ldots, x, y, x, y, \ldots$$

where $x \neq y$. Then,

$$x = \alpha + \beta x + \frac{x}{y}, \tag{3.17a}$$

$$y = \alpha + \beta y + \frac{y}{x}. \tag{3.17b}$$

Subtraction of the latter equation from the former one yields $y = x/[x(1-\beta)-1]$. Plugging this into (3.17a) gives $\alpha = 1$. Notice that $x = 2/(1-\beta)$ results in $y = 2/(1-\beta)$, which contradicts the assumption that $x \neq y$.

Conversely, assume that $\alpha = 1$. Let $x_{-1} \neq \frac{2}{1-\beta}$, $x_{-1} \neq \frac{1}{1-\beta}$ and $x_0 = \frac{x_{-1}}{x_{-1}(1-\beta)-1}$. From (3.1), the following can be deduced:

$$x_1 = 1 + \beta x_{-1} + \frac{x_{-1}}{x_0} = x_{-1}$$

$$x_2 = 1 + \beta x_0 + \frac{x_0}{x_1} = 1 + \beta x_0 + \frac{x_0}{x_{-1}} = x_0.$$

By induction, it is now easy to see that $\{x_n\}_{n=-1}^{\infty}$ is a prime 2-periodic solution. □

Lemma 3.4.2 *Let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of (3.1) which consists of a single semi-cycle. Then,* $\{x_n\}_{n=-1}^{\infty}$ *converges to* $\bar{x} = \frac{1+\alpha}{1-\beta}$.

Proof. Suppose that $\{x_n\}_{n=-1}^{\infty}$ is a positive solution of (3.1) which is a negative semi-cycle. Then, using $1-\beta = (1+\alpha)/\bar{x}$ and $0 < x_n < \bar{x}$, one derives

$$\begin{aligned} x_{2n+2} - x_{2n} &= \alpha + \beta\, x_{2n} + \frac{x_{2n}}{x_{2n+1}} - x_{2n} \\ &= \alpha + \left(\beta - 1 + \frac{1}{x_{2n+1}}\right) x_{2n} \\ &> \alpha + \left(\frac{-1-\alpha}{\bar{x}} + \frac{1}{\bar{x}}\right) x_{2n} \\ &= \alpha - \frac{\alpha}{\bar{x}} x_{2n} \end{aligned}$$

$$= \alpha\left(1 - \frac{x_{2n}}{\bar{x}}\right) \geqslant 0$$

and

$$\begin{aligned}
x_{2n+1} - x_{2n-1} &= \alpha + \beta x_{2n-1} + \frac{x_{2n-1}}{x_2 n} - x_{2n-1} \\
&= \alpha + \left(\beta - 1 + \frac{1}{x_{2n}}\right) x_{2n-1} \\
&> \alpha + \left(\frac{-1-\alpha}{\bar{x}} + \frac{1}{\bar{x}}\right) x_{2n-1} \\
&= \alpha - \frac{\alpha}{\bar{x}} x_{2n-1} \\
&= \alpha\left(1 - \frac{x_{2n-1}}{\bar{x}}\right) \geqslant 0,
\end{aligned}$$

implying that the subsequences $\{x_{2n+1}\}_{n=-1}^{\infty}$ and $\{x_{2n}\}_{n=0}^{\infty}$ are both strictly increasing. Therefore, the limits $\lim_{n\to\infty} x_{2n+1} = L_1$ and $\lim_{n\to\infty} x_{2n} = L_2$ exist. Moreover, $L_1, L_2 \in (0, \bar{x}]$. Since $L_1 = \alpha + \beta L_1 + L_1/L_2$, one has

$$\frac{\alpha}{L_1} + \frac{1}{L_2} = 1 - \beta. \tag{3.18}$$

Now, if $L_1 < \bar{x}$ or $L_2 < \bar{x}$, then $\alpha/L_1 + 1/L_2 > (\alpha + 1)/\bar{x} = 1 - \beta$, which contradicts (3.18). Therefore, $L_1 = L_2 = \bar{x}$ and, hence, $\{x_n\}_{n=-1}^{\infty}$ converges to $\bar{x}$, as claimed.

Now, assume that $\{x_n\}_{n=-1}^{\infty}$ is a positive solution of (3.1) which is a positive semi-cycle. Then, using $1 - \beta = (1 + \alpha)/\bar{x}$ and $x_n \geqslant \bar{x}$, one derives

$$\begin{aligned}
x_{2n+2} - x_{2n} &= \alpha + \beta\, x_{2n} + \frac{x_{2n}}{x_{2n+1}} - x_{2n} \\
&= \alpha + \left(\beta - 1 + \frac{1}{x_{2n+1}}\right) x_{2n} \\
&\leqslant \alpha + \left(\frac{-1-\alpha}{\bar{x}} + \frac{1}{\bar{x}}\right) x_{2n} \\
&= \alpha - \frac{\alpha}{\bar{x}} x_{2n} \\
&= \alpha\left(1 - \frac{x_{2n}}{\bar{x}}\right) \leqslant 0
\end{aligned}$$

and

$$\begin{aligned}
x_{2n+1} - x_{2n-1} &= \alpha + \beta x_{2n-1} + \frac{x_{2n-1}}{x_2 n} - x_{2n-1} \\
&= \alpha + \left(\beta - 1 + \frac{1}{x_{2n}}\right) x_{2n-1}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \alpha + \left(\frac{-1-\alpha}{\bar{x}} + \frac{1}{\bar{x}} \right) x_{2n-1} \\
&= \alpha - \frac{\alpha}{\bar{x}} x_{2n-1} \\
&= \alpha \left(1 - \frac{x_{2n-1}}{\bar{x}} \right) \leqslant 0,
\end{aligned}
$$

implying that the subsequences $\{x_{2n+1}\}_{n=-1}^{\infty}$ and $\{x_{2n}\}_{n=0}^{\infty}$ are both decreasing. Therefore, the limits $\lim_{n\to\infty} x_{2n+1} = L_1$ and $\lim_{n\to\infty} x_{2n} = L_2$ exist. Moreover, $L_1, L_2 \in [\bar{x}, \infty)$. Since $L_1 = \alpha + \beta L_1 + L_1/L_2$, one has

$$
\frac{\alpha}{L_1} + \frac{1}{L_2} = 1 - \beta. \tag{3.19}
$$

Now, if $L_1 > \bar{x}$ or $L_2 > \bar{x}$, then $\alpha/L_1 + 1/L_2 < (\alpha + 1)/\bar{x} = 1 - \beta$, which contradicts (3.19). Therefore, $L_1 = L_2 = \bar{x}$ and, hence, $\{x_n\}_{n=-1}^{\infty}$ converges to $\bar{x}$, as claimed. □

Lemma 3.4.3 *Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of (3.1) which consists of at least two semi-cycles. Then, $\{x_n\}_{n=-1}^{\infty}$ is oscillatory. Moreover, with the possible exception of the first semi-cycle, every semi-cycle has length 1. Aside from that, for any $\varepsilon > 0$, except possibly for finitely many terms, every term of $\{x_n\}_{n=-1}^{\infty}$ is strictly greater than $\frac{\alpha}{1-\beta} - \varepsilon$.*

Proof. Suppose that $\{x_n\}_{n=-1}^{\infty}$ is a positive solution which consists of at least two semi-cycles. Then, there exists $m \geqslant -1$ such that $x_m < \bar{x} \leqslant x_{m+1}$ or $x_{m+1} < \bar{x} \leqslant x_m$. Only the former case will be considered since the latter can be treated similarly. Now,

$$
x_{m+2} = \alpha + \beta x_m + \frac{x_m}{x_{m+1}} < \alpha + \beta\bar{x} + 1 = \bar{x}
$$

and

$$
x_{m+3} = \alpha + \beta x_{m+1} + \frac{x_{m+1}}{x_{m+2}} > \alpha + \beta\bar{x} + 1 = \bar{x}.
$$

It can be shown by induction that

$$
\alpha < x_{m+2k} < \bar{x} \leqslant x_{m+2k+1} \quad \text{for} \quad k \geqslant 0. \tag{3.20}
$$

That is, every semi-cycle, except possibly for the first one, say $\{x_{-1}, \ldots, x_m\}$, has length 1, and the solution $\{x_n\}_{n=-1}^{\infty}$ is oscillatory.

Additionally, by Lemma 3.2.3, it is clear that $\{x_{m+2k}\}_{k=0}^{\infty}$ and $\{x_{m+2k+1}\}_{k=0}^{\infty}$ are eventually monotone. Being bounded and monotone, $x_{m+2k} \to L_1$ as $k \to \infty$, where $\alpha \leqslant L_1 \leqslant \bar{x}$.

In the case when $\{x_{m+2k+1}\}_{k=0}^{\infty}$ is not bounded from above, one has $x_{m+2k+1} \to \infty$ as $k \to \infty$ which, by Lemma 3.3.1, implies that $L_1 = \alpha/(1-\beta)$. On the other hand, if $\{x_{m+2k+1}\}_{k=0}^{\infty}$ is bounded from above, then it has a finite limit, say L_2. It goes without saying that

$$\frac{\alpha}{L_1} + \frac{1}{L_2} = 1 - \beta = \frac{1}{L_1} + \frac{\alpha}{L_2}.$$

Clearly, $L_1 > \alpha/(1-\beta)$ since, otherwise,

$$1 - \beta = \frac{\alpha}{L_1} + \frac{1}{L_2} \geqslant 1 - \beta + \frac{1}{L_2}$$

implies that $L_2 \leqslant 0$, which is a contradiction. Thus, in any case, $L_1 \geqslant \alpha/(1-\beta)$. Using this together with (3.20), one obtains the final result of Lemma 3.4.3. □

Theorem 3.4.4 *Let $\alpha = 1$, $0 \leqslant \beta < 1$, and $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of (3.1). Then, the following statements hold:*

(i) If $\{x_n\}_{n=-1}^{\infty}$ consists of a single semi-cycle, then $\{x_n\}_{n=-1}^{\infty}$ converges to $\bar{x} = \frac{2}{1-\beta}$;

(ii) If $\{x_n\}_{n=-1}^{\infty}$ consists of at least two semi-cycles, then $\{x_n\}_{n=-1}^{\infty}$ converges to a prime 2-periodic solution of (3.1).

Proof. It is known by Lemma 3.4.2 that if $\{x_n\}_{n=-1}^{\infty}$ consists of a single semi-cycle, then $\{x_n\}_{n=-1}^{\infty}$ converges to $\bar{x}$. Otherwise, by Lemma 3.4.3, $\{x_n\}_{n=-1}^{\infty}$ is oscillatory and, except possibly for the first semi-cycle, every semi-cycle has length 1. Now, the proof of the second part follows from Corollary 3.2.2 and Lemma 3.2.4. □

3.5 Stability Analysis

This part of the thesis devoted to the stability analysis of the equilibrium point $\bar{x} = \frac{1+\alpha}{1-\beta}$ of (3.1).

Lemma 3.5.1 *The equilibrium point $\bar{x} = \frac{1+\alpha}{1-\beta}$ of (3.1) is*

(i) locally asymptotically stable if $\alpha > 1$;

(ii) unstable if $0 \leqslant \alpha < 1$.

Proof. The linearized equation of (3.1) about $\bar{x}$ is

$$y_{n+1} = Ay_n + By_{n-1},$$

where $A = -(1-\beta)/(1+\alpha)$ and $B = (1+\alpha\beta)/(1+\alpha)$. Let $0 \leqslant \beta < 1$.

(i) If $\alpha > 1$, then

$$|A| + B - 1 = \frac{(1-\alpha)(1-\beta)}{1+\alpha} < 0 \quad \text{and} \quad 1 - B = \frac{\alpha(1-\beta)}{1+\alpha} < 2,$$

and hence, by Theorem 2.2.8 (iii), $\bar{x}$ is locally asymptotically stable.

(ii) If $0 \leqslant \alpha < 1$, then

$$A^2 + 4B > 0 \quad \text{and} \quad |A| - |1 - B| = \frac{(1-\alpha)(1-\beta)}{1+\alpha} > 0,$$

and hence, by Theorem 2.2.8 (iv), $\bar{x}$ is unstable. □

Lemma 3.5.2 *Let* $\alpha > 1$, *and let* $\{x_n\}_{n=-1}^{\infty}$ *be a positive solution of (3.1). Then,*

$$\frac{\alpha}{1-\beta} + \frac{\alpha-1}{\alpha} \leqslant \liminf_{n\to\infty} x_n \leqslant \limsup_{n\to\infty} x_n \leqslant \frac{\alpha^2}{(\alpha-1)(1-\beta)}.$$

Proof. Because of Lemmas 3.4.2 and 3.4.3, it may be assumed that every semi-cycle of $\{x_n\}_{n=-1}^{\infty}$ has length 1, that $\frac{\alpha}{1-\beta} < x_n$ for all $n \geqslant -1$, and that $\frac{\alpha}{1-\beta} < x_0 < \frac{1+\alpha}{1-\beta} < x_{-1}$. Note that for $n \geqslant 0$,

$$x_{2n+1} = \alpha + \beta x_{2n-1} + \frac{x_{2n-1}}{x_{2n}} < \alpha + \left(\beta + \frac{1-\beta}{\alpha}\right) x_{2n-1}.$$

Thus,

$$x_{2n+1} < \alpha + \alpha\left(\beta + \frac{1-\beta}{\alpha}\right) + \left(\beta + \frac{1-\beta}{\alpha}\right)^2 x_{2n-3}.$$

Successive application of the previous inequality yields

$$x_{2n+1} < \frac{\alpha^2}{(\alpha-1)(1-\beta)}\left[1 - \left(\beta + \frac{1-\beta}{\alpha}\right)^n\right] + \left(\beta + \frac{1-\beta}{\alpha}\right)^n x_{-1}. \tag{3.21}$$

Since $\beta + \frac{1-\beta}{\alpha} < 1$, it follows from (3.20) with $m = 0$ and (3.21) that

$$\limsup_{n\to\infty} x_n \leqslant \frac{\alpha^2}{(\alpha-1)(1-\beta)}.$$

That is, for any $\varepsilon > 0$, there exists $N \geqslant 0$ such that

$$x_{2n+1} < \frac{\alpha^2 + \varepsilon}{(\alpha - 1)(1 - \beta)} \quad \text{for all} \quad n \geqslant N.$$

Thus, for any $n > N$,

$$x_{2n} = \alpha + \beta x_{2n-2} + \frac{x_{2n-2}}{x_{2n-1}} > \alpha + \left(\beta + \frac{(\alpha-1)(1-\beta)}{\alpha^2 + \varepsilon}\right)\frac{\alpha}{1-\beta} = \frac{\alpha}{1-\beta} + \frac{\alpha(\alpha-1)}{\alpha^2+\varepsilon}.$$

Since ε is arbitrary, it follows that

$$\liminf_{n\to\infty} x_n \geqslant \frac{\alpha}{1-\beta} + \frac{\alpha - 1}{\alpha}.$$

□

Theorem 3.5.3 *Let $\alpha > 1$. Then $\bar{x} = \frac{1+\alpha}{1-\beta}$ is a globally asymptotically stable equilibrium point of (3.1).*

Proof. It is known by Lemma 3.5.1 that $\bar{x} = \frac{1+\alpha}{1-\beta}$ is a locally asymptotically stable equilibrium point of (3.1). Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of (3.1). It suffices to show that

$$\lim_{n\to\infty} x_n = \frac{1+\alpha}{1-\beta}$$

For $x, y \in (0, \infty)$, set

$$f(x, y) = \alpha + \beta y + \frac{y}{x}, \quad a = \frac{\alpha}{1-\beta} \quad \text{and} \quad b = \frac{\alpha^2}{(\alpha-1)(1-\beta)}.$$

Then,

$$f(a, b) = \alpha + \frac{\beta\alpha^2}{(\alpha-1)(1-\beta)} + \frac{\alpha}{\alpha - 1} = \frac{\alpha^2}{(\alpha-1)(1-\beta)} = b$$

and

$$f(b, a) = \alpha + \frac{\alpha\beta}{1-\beta} + \frac{\alpha-1}{\alpha} = \frac{\alpha}{1-\beta} + \frac{\alpha-1}{\alpha} > a$$

Hence,

$$a \leqslant f(x, y) \leqslant b \quad \text{for all} \quad x, y \in [a, b].$$

By Lemma 3.4.1, there is no prime 2-periodic solution of (3.1) and, by Theorem 2.2.9,

$$\lim_{n\to\infty} x_n = \frac{1+\alpha}{1-\beta}.$$

□

3.6 Numerical Examples

In the last part of the present chapter some numerical tests are provided to illustrate the theoretical results obtained in this thesis.

Example 3.6.1 *Consider the initial value problem (IVP)*

$$\begin{aligned} x_{n+1} &= 0.2 + 0.5x_{n-1} + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \ldots, \\ x_{-1} &= 1, \quad x_0 = 3. \end{aligned} \tag{3.22}$$

Clearly, the conditions of Theorem 3.3.2 are satisfied and, as a result, $\lim_{n\to\infty} x_{2n} = \infty$ *and* $\lim_{n\to\infty} x_{2n+1} = \alpha/(1-\beta) = 0.4$ *as seen in Figure 3.1.*

Example 3.6.2 *Consider the IVP*

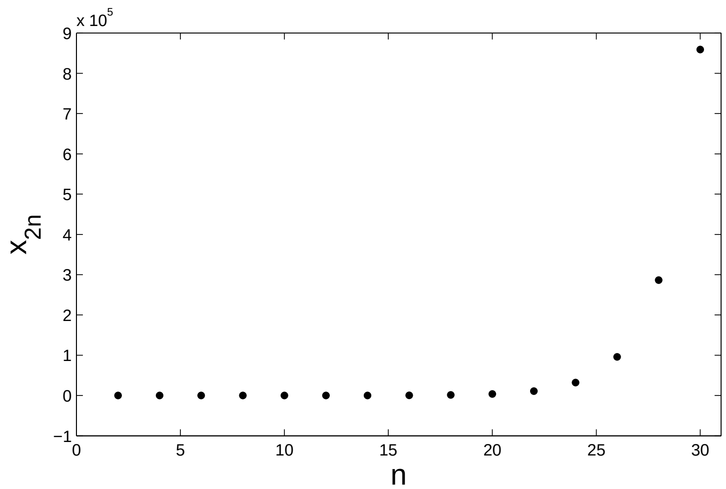
$$\begin{aligned} x_{n+1} &= 1 + 0.5x_{n-1} + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \ldots, \\ x_{-1} &= 1, \quad x_0 = 5. \end{aligned} \tag{3.23}$$

Obviously, the solution $\{x_n\}_{n=-1}^{\infty}$ *of* (3.23) *consists of at least two semi-cycles. Then, by Theorem 3.4.4, this solution converges to a prime 2-periodic solution as per Figure 3.2.*

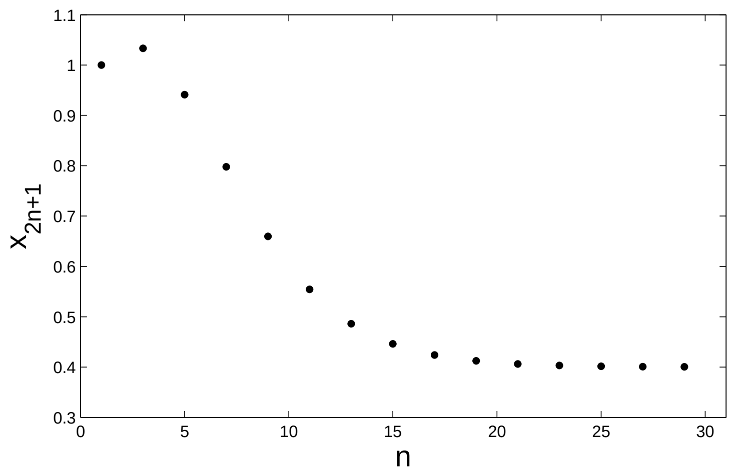
Example 3.6.3 *Consider the IVP*

$$\begin{aligned} x_{n+1} &= 2 + 0.5x_{n-1} + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \ldots, \\ x_{-1} &= 1, \quad x_0 = 3. \end{aligned} \tag{3.24}$$

Since, in this example, $\alpha = 2 > 1$, *by Theorem 3.5.3, the equilibrium point* $\overline{x} = 6$ *of* (3.24) *is globally asymptotically stable. As it can be seen in Figure 3.3, the solution* $\{x_n\}$ *of* (3.24) *converges to the fixed point* $\overline{x} = 6$.



(a) Even index terms



(b) Odd index terms

Figure 3.1: The solution of (3.22).

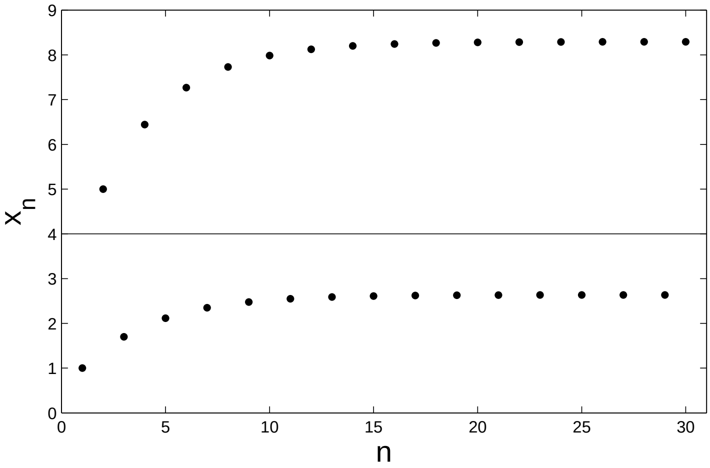


Figure 3.2: The solution of (3.23).

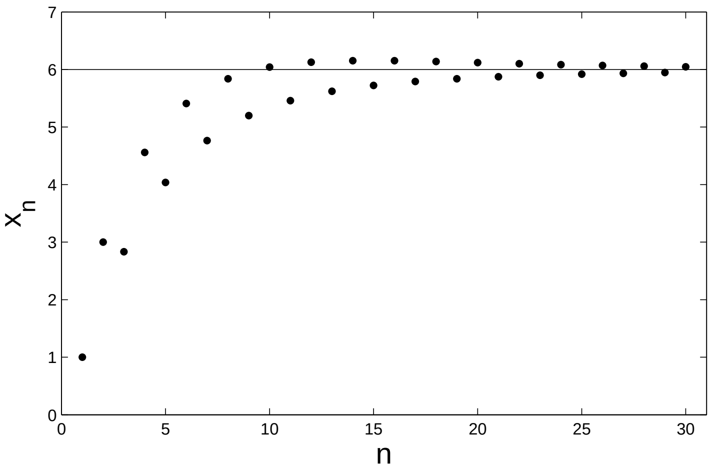


Figure 3.3: The solution of (3.24).

CHAPTER 4

CONCLUSION

In this thesis some dynamics properties of a second order nonlinear difference equation with two real parameters α and β has been studied. The special case $\beta = 0$ has been investigated by many researches during the last two decades. The results obtained in this thesis are more general than those obtained before and if one sets $\beta = 0$ in the present outcomes, then the pioneering results are recovered.

Main results can be categorized in three major classes: boundedness, periodicity and semi-cycle analysis, and stability analysis.

Necessary and sufficient conditions for a positive solution to be bounded or unbounded are derived. It has been shown that a periodic solution can exist only for a special value of α. In this case, the structure of a periodic solution in terms of the initial conditions is given. In addition, semi-cycle analysis of a positive solution is performed. One of the main results about the semi-cycles is that a positive solution consisting of a single semi-cycle must converge to the fixed point, while that consisting of at least two semi-cycles oscillates. Also, in the case when periodic solutions exist, the positive solutions with more than one semi-cycle should converge to prime 2-periodic solution. Moreover, local and global stability analysis of the fixed point are done. The conditions for the equilibrium solution are stated in relation to the parameters involved in the equation. Finally, the obtained results are illustrated by some numerical examples.

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