SERIES SOLUTIONS OF DYNAMIC EQUATIONS ON TIME SCALES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
ATILIM UNIVERSITY

BY

FATMA ALUSTA

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS

SEPTEMBER 2018

Approval of the Graduate School of Natural and Applied Sciences, Atılım University.

Prof. Dr. Ali Kara
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of **Master of Science in Mathematics Department, Atılım University**.

Prof. Dr. Tanıl Ergenç
Head of Department

This is to certify that we have read the thesis SERIES SOLUTIONS OF DYNAMIC EQUATIONS ON TIME SCALES submitted by FATMA ALUSTA and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. İnci Erhan
Supervisor

Examining Committee Members:

Prof. Dr. Fatma Karakoç
Mathematics Department, Ankara University

Assoc. Prof. Dr. Ümit Aksoy
Mathematics Department, Atılım University

Assoc. Prof. Dr. İnci Erhan
Mathematics Department, Atılım University

Date: September 21, 2018

I declare and guarantee that all data, knowledge and information in this document has been obtained, processed and presented in accordance with academic rules and ethical conduct. Based on these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name : FATMA ALUSTA

Signature :

ABSTRACT

SERIES SOLUTIONS OF DYNAMIC EQUATIONS ON TIME SCALES

Alusta, Fatma
M.S., Department of Mathematics
Supervisor : Assoc. Prof. Dr. İnci Erhan

SEPTEMBER 2018, 70 pages

In this thesis, we study the series solution method for dynamic equations on time scales. We propose a series expansion for the solution of a given dynamic equation and derive a very general recurrence relation formula for the computation of the coefficients in this series. The importance of time scales and dynamic equations on time scales shows itself in the fact that time scales unify the continuous and discrete analysis and therefore, dynamic equations cover both the differential and difference equations.

In Chapter 1 we give the definition of time scales, some basic notions on time scales and present some examples. We introduce basic calculus concepts such as delta derivative and integral of function defined on time scales in Chapter 2. In the same chapter we also define some elementary functions on time scales. Chapter 3 is devoted to basic thery of linear dynamic equation of first and higher order. The Series solution method is presented in details in Chapter 4. In Chapter 5 we apply the method to some specific examples of linear dynamic equations including both constant and nonconstant coefficients equations. Finally, we discuss the conclussion in Chapter 6.

Keywords: Time scales, dynamic equation, Delta derivative, series solution

ÖZ

ZAMAN SKALASINDA DİNAMİK DENKLEMLERİN SERİ ÇÖZÜMLERİ

Alusta, Fatma

Yüksek Lisans, Matematik Bölümü

Tez Yöneticisi : Doç. Dr. İnci Erhan

Eylül 2018, 70 sayfa

Bu tez çalışmasında, zaman skalasında dinamik denklemler için seri çözüm yöntemini çalıştık. Verilen bir dinamik denklemin çözümü için seri açılımı önerdik ve bu serinin katsayılarını belirlemek için genel bir rekürans bağıntısı elde ettik. Zaman skalası ve dinamik denklemlerin önemi, zaman skalasının, sürekli ve kesikli analizi birleştirmesinde ve dolayısıyla dinamik denklemler de, differansiyel ve fark denklemlerini kapsamasında kendini belli etmektedir.

Bölüm 1'de zaman skalası ve bazı ilgili kavramların tanımları ile birlikte örnekler verdik. Zaman skalasında tanımlı fonksiyonlar için Delta türev ve integral gibi temel analiz kavramlarını Bölüm 2'de verdik. Bu bölümde aynı zamanda bazı elemanter fonksiyonları da tanımladık. Üçüncü bölüm birinci ve daha yüksek mertebeden dinamik denklemlerin temel teorisine adanmıştır. Seri çözüm yöntemi ayrıntılı olarak Bölüm 4'de açıklanmıştır. Bölüm 5'de bu yöntemi, sabit ve değişken katsayılı olmak üzere belirli doğrusal dinamik denklem örneklerine uyguladık. Son olarak, Bölüm 6'da sonuçları tartıştık.

Anahtar Kelimeler: Zaman skalası, dinamik denklem, Delta türev, seri çözüm

To my dearest parents, my husband and my lovely children

ACKNOWLEDGMENTS

In the beginning, praise be to Allah for his grace which is countless.

Secondly I shall also thank my country Libya for providing me with the financial support for my graduate studies.

I would like to express my gratitude to my supervisor Assoc. Prof. Dr. İnci Erhan for her knowledge, time and advices and sparing no effort to help me. She has all the thanks and gratitude.

Also, I would like to thank the distinguisshed members of the examining committe Assoc. Prof. Dr. Ümit Aksoy and Prof. Dr. Fatma Karakoç for their valuable comments and suggestions on the thesis. I also want to express my thanks and appreciation to all my teachers.

My special thanks to my dear mother for her being the fountain of patience and optimism, to the big heart of my dear father, my dear husband, my lovely children and all my family.

Finally, my thanks go to everyone who stayed with me and supported me throught my studies.

TABLE OF CONTENTS

LIST OF SYMBOLS

$\mathbb{N}$	:	the set of natural numbers
$\mathbb{N}_0$	:	the set of non-negative integers
$\mathbb{Z}$	:	the set of integers
$\mathbb{R}$	:	the set of real numbers
$\mathbb{T}$	:	a time scale
Δ	:	Delta derivative on time scale

CHAPTER 1

INTRODUCTION

The concept of time scales has been introduced for the first time by Stefan Hilger [28] in 1988. In his PhD thesis he defined the notion of measure chains and aimed to unify the discrete and continuous analysis. Time scales can be regarded as a special case of measure chains and starting with the pioneering work of Hilger, this subject has received great attention in the recent years [16, 17, 1, 10, 30, 19, 20]. The main advantage of studying time scales is that one can generalize the cases of continuous and discrete models and thus, avoid studying the subject twice. From this point of view, time scales are very appropriate when modeling problem in which the quantities can vary from continuous to discrete. After the basic introduction by Hilger [29], many authors contributed to the analysis on time scales [6, 5, 9, 29, 32, 33, 11, 18].

Definition 1.0.1 *[16] A time scale is a nonempty closed subset of the set of real numbers $\mathbb{R}$.*

For example, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$, the closed interval $[1, 3]$ are time scales, but $(1, 2)$, $[1, 3)$ or $(-1, 0]$ are not time scales.

The most commonly used notation for a time scale is $\mathbb{T}$.
We define next some basic concepts on time scales.

Definition 1.0.2 *[16] Let $\mathbb{T}$ be a time scale.*

(1) Forward jump operator $\sigma(t) : \mathbb{T} \to \mathbb{T}$ is defined as

$$\sigma(t) = \inf\{s \in \mathbb{T} | s > t\}, \quad t \in \mathbb{T}. \tag{1.1}$$

(2) Backward jump operator $\rho(t) : \mathbb{T} \to \mathbb{T}$ is defined as

$$\rho(t) = \sup\{s \in \mathbb{T} | s < t\}, \quad t \in \mathbb{T}. \tag{1.2}$$

(3) Graininess function $\mu(t) : \mathbb{T} \to [0, \infty)$ is defined as

$$\mu(t) = \sigma(t) - t, \quad t \in \mathbb{T}. \tag{1.3}$$

Remark 1.0.3 *Notice that $\sigma(t) \geq t \quad$ and $\quad \rho(t) \leq t,\;$ for all $\; t \in \mathbb{T}$.*

Remark 1.0.4 *We assume that $\sup \emptyset = \inf \mathbb{T}, \inf \emptyset = \sup \mathbb{T}$, where $\emptyset$ denotes the empty set.*

Definition 1.0.5 *[16] Let $\mathbb{T}$ be a time scale with forward jump and backward jump operators σ and ρ respectively. Then*

- *If $\sigma(t) = t$, then t is right-dense.*
- *If $\sigma(t) > t$, then t is right-scattered.*
- *If $\rho(t) = t$, then t is left-dense.*
- *If $\rho(t) < t$, then t is left-scattered.*
- *If t is left and right-scattered, then t is isolated.*

The following examples illustrate these concepts on various time scales.

Example 1.0.6 *[26] Let $\mathbb{T} = 2\mathbb{N}_0 = \{0, 2, 4, 6, 8, ...\}$.*
For any $t \in \mathbb{T}, \quad t = 2n$ and we compute

$$\begin{aligned} \sigma(t) &= \inf\{s \in 2\mathbb{N}_0 \mid s > t\} = \inf\{2l \mid 2l > 2n\} \\ &= 2(n + 1) = 2n + 2 = t + 2. \end{aligned}$$

Then we have

$$\sigma(t) = t + 2.$$

To compute $\rho(t)$, we consider separately the following cases:

For $t = 0$ we compute $\rho(0) = \sup\{2l \mid 2l < 0\} = \sup \emptyset = \inf \mathbb{T} = 0$.

For $t > 0$ we have $t = 2n$ where $n \geq 1$ and

$$\rho(t) = \sup\{2l \mid 2l < 2n\} = 2(n-1) = 2n - 2.$$

Then

$$\rho(t) = \begin{cases} t-2 & if \quad t \neq 0 \\ 0 & if \quad t = 0 \end{cases}.$$

and

$$\sigma(t) = t + 2, \quad \mu(t) = \sigma(t) - t = t + 2 - t = 2.$$

Example 1.0.7 *[26] Let*

$$\mathbb{T} = 2^{\mathbb{N}_0} = \{2^0, 2^1, 2^3, \ldots\} = \{1, 2, 4, 8, \ldots\}$$

If $t \in \mathbb{T}$, then $t = 2^n$. We compute

$$\sigma(t) = \inf\{2^l \mid 2^l > 2^n\} = 2^{n+1} = 2.2^n = 2t.$$

On the other hand,

$$\rho(1) = \sup\{2^l \mid 2^l < 1\} = \sup \emptyset = \inf \mathbb{T} = 1,$$

and if $t \neq 1$

$$\rho(t) = \sup\{2^l \mid 2^l < 2^n\} = 2^{n-1} = \frac{2^n}{2} = \frac{t}{2}.$$

Then we have

$$\sigma(t) = 2t,$$

$$\rho(t) = \begin{cases} \dfrac{t}{2} & if \quad t \neq 1 \\ 1 & if \quad t = 1 \end{cases},$$

and

$$\mu(t) = \sigma(t) - t = 2t - t = t.$$

Notice that

- For $t \in \mathbb{T}$, $\sigma(t) = 2t > t$, so t is right-scattered,

- For $t \neq 1, \rho(t) = \frac{1}{2}t < t$, so t is left-scattered,

- For $t = 1, \rho(1) = 1$, so that 1 is left-dense.

Example 1.0.8 *[26] Let* $\mathbb{T} = \left\{\frac{1}{n}, n \in \mathbb{N}\right\} \cup \{0\} = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$.
Here $t = \frac{1}{n}$ *and* $n = \frac{1}{t}$. *We compute first the forward jump operator* $\sigma(t)$.
For $t \neq 0$ *we compute* $\sigma(t) = \inf\left\{0, \frac{1}{l} \,\middle|\, \frac{1}{l} > \frac{1}{n}\right\} = \frac{1}{n-1} = \frac{1}{\frac{1}{t} - 1} = \frac{t}{1-t}$,
for $t = 0$ *we have* $\sigma(0) = \inf\left\{\frac{1}{l}, \frac{1}{l} > 0\right\} = \inf \mathbb{T} = \sup \emptyset = 0$,
and for $t = 1$ *we have* $\sigma(1) = \inf\left\{\frac{1}{l}, \frac{1}{l} > 1\right\} = \inf \emptyset = \sup \mathbb{T} = 1$. *Hence,*

$$\sigma(t) = \begin{cases} \frac{t}{1-t} & if \quad t \neq 1 \\ 1 & if \quad t = 1 \\ 0 & if \quad t = 0 \end{cases}.$$

Now we compute $\rho(t)$.
For $t \neq 0$*,we have* $\rho(t) = \sup\left\{\frac{1}{l} \,\middle|\, \frac{1}{l} < \frac{1}{n}\right\} = \frac{1}{n+1} = \frac{1}{\frac{1}{t} + 1} = \frac{t}{1+t}$,
and for $t = 0$ *we have* $\rho(0) = \sup\left\{\frac{1}{l}, \frac{1}{l} < 0\right\} = \sup \emptyset = \inf \mathbb{T} = 0$. *Then*

$$\rho(t) = \begin{cases} \frac{t}{1+t} & if \quad t \neq 0 \\ 0 & if \quad t = 0 \end{cases}.$$

Finally we have

$$\mu(t) = \sigma(t) - t = \begin{cases} \frac{t}{1-t} - t & if \quad t \neq 0, t \neq 1 \\ 0 & if \quad t = 0 \\ 0 & if \quad t = 1. \end{cases}$$

In this example we notice that

- $t \in \mathbb{T}, t \neq 0, t \neq 1$ *is right-scattered and left-scattered,*

- $t = 0$ *is right-dense and left-dense,*

- $t = 1$ *is right-dense and left-scattered.*

Example 1.0.9 *[26] Let* $\mathbb{T} = \left\{\sqrt{2n+1}, n \in \mathbb{T}\right\} = \{\sqrt{3}, \sqrt{5}, \sqrt{7}, \ldots\}$*. Then we have* $t = \sqrt{2n+1}$*, so that* $n = \dfrac{t^2-1}{2}$.

We compute

$$\begin{aligned}\sigma(t) &= \inf\{\sqrt{2l+1} | \sqrt{2l+1} > \sqrt{2n+1}\}\\ &= \sqrt{2(n+1)+1}\\ &= \sqrt{2n+3}\\ &= \sqrt{t^2+2}.\end{aligned}$$

For $t \neq \sqrt{3}$,

$$\begin{aligned}\rho(t) &= \sup\{\sqrt{2l+1} | \sqrt{2l+1} < \sqrt{2n+1}\}\\ &= \sqrt{2(n-1)+1}\\ &= \sqrt{2n-1}\\ &= \sqrt{t^2-2}.\end{aligned}$$

For $t = \sqrt{3}$,

$$\begin{aligned}\rho(\sqrt{3}) &= \sup\{\sqrt{2l+1} | \sqrt{2l+1} < \sqrt{3}\}\\ &= \sup \emptyset\\ &= \inf \mathbb{T}\\ &= \sqrt{3}.\end{aligned}$$

Then we obtain

$$\sigma(t) = \sqrt{t^2-2},$$

and

$$\rho(t) = \begin{cases} \sqrt{t^2-2} & \text{if} \quad t \neq \sqrt{3} \\ \sqrt{3} & \text{if} \quad t = \sqrt{3} \end{cases}.$$

Our last example is a time scales which models biology problems related with the life span of certain species[17].

Example 1.0.10 *Let*

$$\mathbb{T} = P_{a,b} = \bigcup_{k=0}^{\infty}[k(a+b), k(a+b)+a] = [0,a]\cup[a+b, 2a+b]\cup[2(a+b), 3a+2b]\cup...., \quad a > b > 0.$$

Then we have

$$\sigma(t) = \begin{cases} t & \text{if} \quad t \in \bigcup_{k=0}^{\infty}[k(a+b), k(a+b)+a) \\ t+b & \text{if} \quad t = k(a+b)+a \end{cases}$$

and

$$\mu(t) = \begin{cases} 0 & if \quad t \in \bigcup_{k=0}^{\infty}[k(a+b), k(a+b)+a) \\ b & if \quad t = k(a+b)+a \end{cases}$$

In particular if we take $a = b = 1$ we obtain

$$P_{1,1} = \bigcup_{k=0}^{\infty}[2k, 2k+1] = [0,1] \cup [2,3] \cup [4,5] \cup ....$$

This example models the life span of a certain insect species in one unit of time. Suppose that just before the insect dies out, eggs are laid which are hatched one unit of time later. For this time scale,

$$\mu(t) = \begin{cases} 0 & if \quad t \in \bigcup_{k=0}^{\infty}[2k, 2k+1) \\ 1 & if \quad t = 2k+1 \end{cases} .$$

For a specific example of this type we can mention the insect known as magicicada septendecim whic lives as a larva for 17 years and as an adult for a week and also the common mayfly which lives as a larva for a year and as an adult for less than a day.

In this thesis we study a series solution method for dynamic equations on time scales. Dynamic equations can be regarded as a generalization of differential equations on arbitrary time scales. If the time scales under consideration is $\mathbb{T} = \mathbb{R}$, then the dynamic equation becomes a differential equation.

This thesis is organized as follows. In Chapter 2 we introduce the notions of basic calculus on time scales such as delta differentiation, integration, some elementary functions and Taylor series. In Chapter 3 we give the definitions and solutions of linear dynamic equations of first order and constant cofficient equations of higher order. In Chapter 4 we discuss the series solution method for a general n-th order linear dynamic equation. We apply the method to particular examples in Chapter 5 and finally, make a conclussion in Chapter 6.

CHAPTER 2

CALCULUS ON TIME SCALES

In this chapter we introduce some basic concepts of calculus on time scales, such as delta differentiation, integration and some elementary functions. These notions have been initially defined by Hilger [28]. Later, in a series of papers and monographs a complete and rigorous calculus on both one and multi-variable time scales was established [5, 16, 17, 1, 31, 16, 10, 34, 30, 19, 20, 21]. We also define the monomials and discuss their properties and using these basic functions,we give the Taylor series expansion and Taylor's formula on time scales.

2.1 Delta derivative on time scales

We start with the definition, properties and examples of delta derivative of functions defined on time scales.

Definition 2.1.1 *[16, 26] Let $\mathbb{T}$ be a time scale. We define the set $\mathbb{T}^\kappa$ as follows*

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}) & \textit{if} \quad \sup\mathbb{T} < \infty \\ \mathbb{T}, & \textit{if} \quad \sup\mathbb{T} = \infty \end{cases}.$$

The delta derivative of a function is defined next.

Definition 2.1.2 *[16, 26] Let $f : \mathbb{T} \to \mathbb{R}$, be a function and $t \in \mathbb{T}$.*
We define the delta derivative of f at t as a function $f^\Delta(t)$ for which, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \ \textit{ for all } s \in U,$$

where $U = (t - \delta, t + \delta)$.

The following theorem is proved in [16].

Theorem 2.1.3 *Let* $f : \mathbb{T} \to \mathbb{R}, \quad t \in \mathbb{T}^k$. *Then we have*

(1) If f is delta differentiable at t, then f is continuous at t.

(2) If f is continuous at t and t is right-scattered ($\sigma(t) > t$*), then f is delta differentiable at t with*

$$f^\Delta(t) = \frac{f((\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{f((\sigma(t)) - f(t)}{\mu(t)}.$$

(3) If t is right-dense ($\sigma(t) = t$*), then f is delta differentible at t with*

$$f^\Delta(t) = f'(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(4) If f is delta differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

We compute the delta derivatives of some functions in the following examples.

Example 2.1.4 *[26] Let* $f : \mathbb{T} \to \mathbb{R}$ *be defined as* $f(t) = t$. *Then* $f^\Delta(t) = 1$.
To show this we consider the right-scattered and right-dense points t separately.

(1) If t is right-scattered :

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{\sigma(t) - t}{\sigma(t) - t} = 1.$$

(2) If t is right-dense :

$$f^\Delta(t) = f'(t) = 1.$$

Example 2.1.5 *[26] Let* $f : \mathbb{T} \to \mathbb{R}$ *be given as* $f(t) = t^2$. *Then* $f^\Delta(t) = t + \sigma(t)$.
We compute $f^\Delta(t)$ *as follows:*

(1) If t is right-scattered :

$$\begin{aligned} f^{\Delta}(t) &= \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{(\sigma(t))^2 - t^2}{\sigma(t) - t} \\ &= \frac{(\sigma(t) - t)(\sigma(t) + t)}{(\sigma(t) - t)} = \sigma(t) + t. \end{aligned}$$

(2) If t is right-dense :

$$f^{\Delta}(t) = f'(t) = 2t = t + t = \sigma(t) + t.$$

Example 2.1.6 *[26] For $f(t) = \sqrt{t}$, on any time scales $\mathbb{T}$ we have $f^{\Delta}(t) = \dfrac{1}{\sqrt{t} + \sqrt{\sigma(t)}}$. Again, consider the cases with right-scattered and right-dense points separately.*

(1) If t is right-scattered :

$$\begin{aligned} f^{\Delta}(t) &= \frac{\sqrt{\sigma(t)} - \sqrt{t}}{\sigma(t) - t} = \frac{(\sqrt{\sigma(t)} - \sqrt{t})(\sqrt{\sigma(t)} + \sqrt{t})}{(\sigma(t) - t)(\sqrt{\sigma(t)} + \sqrt{t})} \\ &= \frac{(\sigma(t) - t)}{(\sigma(t) - t)(\sqrt{\sigma(t)} + \sqrt{t})} = \frac{1}{\sqrt{t} + \sqrt{\sigma(t)}}. \end{aligned}$$

(2) If t is right-dense :

$$\begin{aligned} f^{\Delta}(t) &= \lim_{s \to t} \frac{\sqrt{t} - \sqrt{s}}{t - s} = \lim_{s \to t} \frac{(\sqrt{t} - \sqrt{s})(\sqrt{t} + \sqrt{s})}{(t - s)(\sqrt{t} + \sqrt{s})} \\ &= \lim_{s \to t} \frac{t - s}{(t - s)(\sqrt{t} + \sqrt{s})} = \frac{1}{2\sqrt{t}} \\ &= \frac{1}{\sqrt{t} + \sqrt{\sigma(t)}}. \end{aligned}$$

Example 2.1.7 *[16] Let $f : \mathbb{T} \to \mathbb{R}$ be given as $f(t) = t^3$.Then $f^{\Delta}(t) = \sigma^2(t) + t\sigma(t) + t^2$.*

(1) If t is right-scattered :

$$\begin{aligned} f^{\Delta}(t) &= \frac{(\sigma(t))^3 - (t)^3}{\sigma(t) - t} \\ &= \frac{(\sigma(t) - t)((\sigma(t))^2 + t\sigma(t) + t^2)}{\sigma(t) - t} \\ &= \sigma^2(t) + t\sigma(t) + t^2. \end{aligned}$$

(2) If t is right-dense :

$$\begin{aligned} f^{\Delta}(t) &= \lim_{s\to t}\frac{t^3-s^3}{t-s} \\ &= \lim_{s\to t}\frac{(t-s)(t^2+ts+s^2)}{(t-s)} \\ &= \lim_{s\to t} t^2+ts+s^2 \\ &= t^2+t^2+t^2 \\ &= \sigma^2(t)+t\sigma(t)+t^2. \end{aligned}$$

The following examples are related with computation of delta derivative on particular time scales.

Example 2.1.8 *Let* $\mathbb{T}=\left\{\frac{1}{2n+1} : n\in\mathbb{N}_0\right\}\cup\{0\}$, *and* $f(t)=\sigma(t)$. *Find* $f^{\Delta}(t)$.
When $t\neq 0, t\neq 1$, *we have* $t=\frac{1}{2n+1}$, *so* $n=\frac{1-t}{2t}$

$$\sigma(t)=\inf\left\{\frac{1}{2l+1},0 \;\middle|\; \frac{1}{2l+1}>\frac{1}{2n+1}\right\}=\frac{1}{2n-1}.$$

Then

$$\sigma(t)=\frac{1}{2(\frac{1-t}{2t})-1}=\frac{t}{1-2t}>t.$$

If $t=0$

$$\sigma(0)=\inf\left\{\frac{1}{2l+1} \;\middle|\; \frac{1}{2l+1}>0\right\}=\inf\mathbb{T}=0.$$

If $t=1$

$$\sigma(1)=\inf\left\{\frac{1}{2l+1},0 \;\middle|\; \frac{1}{2l+1}>1\right\}=\inf\emptyset=\sup\mathbb{T}=1.$$

Notice that t=0 and t=1 are right-dense.
For $t\neq 0,\quad t\neq 1$ *we compute*

$$\begin{aligned} f^{\Delta}(t)=\sigma^{\Delta}(t) &= \frac{\sigma(\sigma(t))-\sigma(t)}{\sigma(t)-t}=\frac{(\frac{\sigma(t)}{1-2\sigma(t)})-\sigma(t)}{\sigma(t)-t} \\ &= \frac{\sigma(t)-\sigma(t)+2(\sigma(t))^2}{(1-2\sigma(t))(\sigma(t)-t)}=\frac{2(\frac{t}{1-2t})^2}{(1-\frac{2t}{1-2t})(\frac{t}{1-2t}-t)} \\ &= \frac{2t^2}{2t^2(1-4t)}=\frac{1}{1-4t}. \end{aligned}$$

For $t = 0$

$$\sigma^{\Delta}(0) = \lim_{s\to 0^+} \frac{\sigma(s)-\sigma(0)}{s-0} = \lim_{s\to 0^+} \frac{\frac{s}{1-2s}-0}{s} = \lim_{s\to 0^+} \frac{s}{s(1-2s)} = 1.$$

For $t = 1$

$$\sigma^{\Delta}(1) = \lim_{s\to 1^-} \frac{\sigma(s)-\sigma(1)}{s-1} = \lim_{s\to 1^-} \frac{\frac{s}{1-2s}-1}{s-1} = \lim_{s\to 1^-} \frac{\frac{s-1+2s}{1-2s}}{s-1} = \lim_{s\to 1^-} \frac{3s-1}{(1-2s)(s-1)} = \infty.$$

In this example $\sigma^{\Delta}(0) = 1$, but $\sigma^{\Delta}(1)$ does not exist!

Example 2.1.9 *[26] Let $\mathbb{T} = \{n^2 : n \in \mathbb{N}_0\}$, $t \in \mathbb{T}$, $f(t) = t^2$and $g(t) = \sigma(t)$. Compute $f^{\Delta}(t)$ and $g^{\Delta}(t)$.*
For $t \in \mathbb{T}$, $t = n$, $n = \sqrt{t}$ and we have

$$\sigma(t) = \inf\{l^2 \mid l^2 > n^2\} = (n+1)^2 = (\sqrt{t}+1)^2 = t + 2\sqrt{t} + 1 > t.$$

We know that $(t^2)^{\Delta} = t + \sigma(t)$ on any time scales. Then

$$f^{\Delta}(t) = (t^2)^{\Delta} = t + \sigma(t) = t + (\sqrt{t}+1)^2 = t + t + 2\sqrt{t} + 1 = 2t + 2\sqrt{t} + 1.$$

Now we will find $g^{\Delta}(t)$

$$\begin{aligned} g^{\Delta}(t) &= (\sigma(t))^{\Delta} \\ &= \frac{\sigma(\sigma(t)) - \sigma(t)}{\sigma(t) - t} \\ &= \frac{\sigma((\sqrt{t}+1)^2) - (\sqrt{t}+1)^2}{(\sqrt{t}+1)^2 - t} \\ &= \frac{\sigma(t + 2\sqrt{t} + 1) - t - 2\sqrt{t} - 1}{t + 2\sqrt{t} + 1 - t} \\ &= \frac{(\sqrt{t + 2\sqrt{t} + 1} + 1)^2 - t - 2\sqrt{t} - 1}{1 + 2\sqrt{t}} \\ &= \frac{2\sqrt{t + 2\sqrt{t} + 1} + 1}{1 + 2\sqrt{t}} \\ &= \frac{2\sqrt{(\sqrt{t}+1)^2} + 1}{1 + 2\sqrt{t}} \\ &= \frac{3 + 2\sqrt{t}}{1 + 2\sqrt{t}}. \end{aligned}$$

Computation of delta derivaties can be simplified by using the differentiation rules as with the ordinary derivatives. The rules are given in the following theorem.

Theorem 2.1.10 *[16] Let $f, g : \mathbb{T} \to \mathbb{R}$ be delta differentiable at $t \in \mathbb{T}^k$. Then*

(1) $(f \pm g)^{\Delta}(t) = f^{\Delta}(t) \pm g^{\Delta}(t)$.

(2) For any constant $\alpha \in \mathbb{R}, \quad (\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t)$.

(3) If $f(t)f(\sigma(t)) \neq 0$ *then*

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

(4) If $g(t)g(\sigma(t)) \neq 0$ *then*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t).g(t) - f(t)g^{\Delta}(t)}{g(t).g(\sigma(t))}.$$

(5)

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + g(\sigma(t))f^{\Delta}(t).$$

Remark 2.1.11 *The function $f(\sigma(t))$ is denoted as $f^{\sigma}(t)$ for brevity.*

Example 2.1.12 *[26] Let $f : \mathbb{T} \to \mathbb{R}$ be differentiable at $t \in \mathbb{T}^k$.Then*

$$(f^n)^{\Delta}(t) = f^{\Delta}(t)\sum_{k=0}^{n-1} f^k(t)(f^{\sigma})^{n-1-k}(t).$$

We will prove the identity using induction.
First, note that for $n = 1$ we have $f^{\Delta}(t) = f^{\Delta}(t)$ and for $n = 2$

$$\begin{aligned}(f^2(t))^{\Delta} &= (f(t)f(t))^{\Delta} \\ &= f^{\Delta}(t)f(t) + f(\sigma(t))f^{\Delta}(t) \\ &= f^{\Delta}(t)(f(t) + f^{\sigma}(t)).\end{aligned}$$

Now assume that

$$(f^n)^\Delta(t) = f^\Delta(t)\sum_{k=0}^{n-1} f^k(t)(f^\sigma)^{n-1-k}(t)$$

for some $n \in \mathbb{N}$*. Then for* $n+1$*, we have*

$$\begin{aligned}
(f^{n+1})^\Delta(t) = (f.f^n)^\Delta(t) &= f^\Delta(t)f^n(t) + f(\sigma(t))(f^n)^\Delta \\
&= f^\Delta(t)f^n(t) + f(\sigma(t))\left(f^\Delta(t)\sum_{k=0}^{n-1} f^k(t)(f^\sigma)^{n-1-k}(t)\right) \\
&= f^\Delta(t)\left(f^n(t) + f(\sigma(t))\sum_{k=0}^{n-1} f^k(t)(f^\sigma)^{n-1-k}(t)\right) \\
&= f^\Delta(t)\left(f^n(t) + \sum_{k=0}^{n-1} f^k(t)(f^\sigma)^{n-k}(t)\right) \\
&= f^\Delta(t)\left((f^\sigma)^n + f(f^\sigma)^{n-1} + ... + f^{n-1}f^\sigma + f^n\right) \\
&= f^\Delta(t)\sum_{k=0}^{n} f^k(t)(f^\sigma)^{n-k}(t).
\end{aligned}$$

Example 2.1.13 *[26] Let* $f(t) = (t-a)^m$ $, a \in \mathbb{R},\ m \in \mathbb{N}$*. Let* $h(t) = t - a$*. Then* $h^\Delta(t) = 1$ *and*

$$f(t) = (h(t))^m.$$

We compute

$$\begin{aligned}
f^\Delta(t) &= ((h(t))^m)^\Delta \\
&= h^\Delta(t).\sum_{k=0}^{m-1}(h(t))^k.(h(\sigma(t))^{m-1-k} \\
&= \sum_{k=0}^{m-1}(t-a)^k(\sigma(t)-a)^{m-1-k}
\end{aligned}$$

Example 2.1.14 *[26] For any function* $f : \mathbb{T} \to \mathbb{R}$ *we have*

$$f^\sigma(t) = f(t) + \mu(t)f^\Delta(t) \ \text{ and } \ f^{\sigma\Delta}(t) = (1 + \mu^\Delta(t))f^{\Delta\sigma}(t).$$

From

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f^\sigma(t) - f(t)}{\mu(t)},$$

we obtain $f^\sigma(t) = f(t) + \mu(t)f^\Delta(t)$ *directly if t is right-scattered. If t is right-dense, then* $\mu(t) = 0$ *and* $f^\sigma(t) = f(\sigma(t)) = f(t)$*.*

For the second identity, we have

$$\begin{aligned} f^{\Delta\sigma}(t) &= \left(\frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}\right)^{\sigma} \\ &= \frac{f(\sigma(\sigma(t))) - f(\sigma(t))}{\sigma(\sigma(t)) - \sigma(t)} \\ &= \frac{f(\sigma(\sigma(t))) - f(\sigma(t))}{\sigma(t) - t} . \frac{\sigma(t) - t}{\sigma(\sigma(t)) - \sigma(t)} \\ &= \frac{f^{\sigma}(\sigma(t)) - f^{\sigma}(t)}{\sigma(t) - t} . \frac{1}{\frac{\sigma(\sigma(t)) - \sigma(t)}{\sigma(t) - t}} \\ &= (f^{\sigma})^{\Delta}(t) . \frac{1}{\sigma^{\Delta}(t)} \\ &= f^{\sigma\Delta}(t) . \frac{1}{(t + \mu(t))^{\Delta}} \\ &= f^{\sigma\Delta}(t) . \frac{1}{1 + \mu^{\Delta}(t)}, \end{aligned}$$

so that we conclude

$$f^{\sigma\Delta}(t) = (1 + \mu^{\Delta}(t)) f^{\Delta\sigma}(t).$$

Next we give some theorems regarding delta derivative including the Mean Value Theorem on time scales.

Theorem 2.1.15 *[16] Let f have Δ derivative at each point of $[a, b]$. If $f(a) = f(b)$, then there exists $\xi_1, \xi_2 \in [a, b]$ such that*

$$f^{\Delta}(\xi_2) \leq 0 \leq f^{\Delta}(\xi_1).$$

Proof.

Since f is delta differentible at each point of $[a, b]$, then f is continuous on $[a, b]$. Therefore f has maximum and minimum values on $[a, b]$. Let

$$m = \min f(t) = f(\xi_1), M = \max f(t) = f(\xi_2).$$

Since $f(a) = f(b)$ then $\xi_1, \xi_2 \in [a, b), ($or in$(a, b])$.
We consider the following cases:

(1) ξ_1 is right-scattered. Then we have

$$f^{\Delta}(\xi_1) = \frac{f(\sigma(\xi_1)) - f(\xi_1)}{\sigma(\xi_1) - \xi_1} \geq 0.$$

(2) ξ_1 is right-dense. Then we compute

$$f^{\Delta}(\xi_1) = \lim_{t \to \xi_1} \frac{f(t) - f(\xi_1)}{t - \xi_1} \geq 0.$$

(3) ξ_2 is right-scattered. Then we have

$$f^{\Delta}(\xi_2) = \frac{f(\sigma(\xi_2)) - f(\xi_2)}{\sigma(\xi_2) - \xi_2} \leq 0.$$

(4) ξ_2 is right-dense. Then

$$f^{\Delta}(\xi_2) = \lim_{t \to \xi_2} \frac{f(t) - f(\xi_2)}{t - \xi_2} \leq 0,$$

so the proof of the theorem is completed.

Theorem 2.1.16 *[16] Let f be delta differentiable at t_0. If $f^{\Delta}(t_0) > 0$, then for some $\delta > 0$*

$$f(\sigma(t)) \geq f(t_0) \quad \forall t \in (t_0, t_0 + \delta)$$

and

$$f(\sigma(t)) \leq f(t_0) \quad \forall t \in (t_0 - \delta, t_0).$$

If $f^{\Delta}(t_0) < 0$, then for some $\delta > 0$

$$f(\sigma(t)) \leq f(t_0) \quad \forall t \in (t_0, t_0 + \delta)$$

and

$$f(\sigma(t)) \geq f(t_0) \quad \forall t \in (t_0 - \delta, t_0).$$

Theorem 2.1.17 *(Mean Value Theorem)[26]*
Let f be continuous on $[a, b]$ and has delta derivative on $[a, b)$. Then there exist ξ_1,$\xi_2 \in [a, b)$ such that

$$f^{\Delta}(\xi_1)(b - a) \leq f(b) - f(a) \leq f^{\Delta}(\xi_2)(b - a).$$

Proof.
Consider

$$\phi(t) = f(t) - f(a) - \frac{f(b) - f(a)}{b - a}(t - a).$$

Then $\phi(t)$ is continuous on $[a, b]$ and has delta derivative on $[a, b)$. From Theorem 2.1.15, there exist ξ_1, ξ_2 such that

$$\phi^{\Delta}(\xi_1) \leq 0 \leq \phi^{\Delta}(\xi_2)$$

and

$$\phi^{\Delta}(t) = f^{\Delta}(t) - \frac{f(b) - f(a)}{b - a}.$$

Then we obtain

$$\phi^{\Delta}(\xi_1) = f^{\Delta}(\xi_1) - \frac{f(b) - f(a)}{b - a} \le 0 \le f^{\Delta}(\xi_2) - \frac{f(b) - f(a)}{b - a} = \phi^{\Delta}(\xi_2),$$

or

$$f^{\Delta}(\xi_1) \le \frac{f(b) - f(a)}{b - a} \le f^{\Delta}(\xi_2),$$

which completes the proof.

Corollary 2.1.18 *[26] Let f be continuous on $[a, b]$ and has delta derivative on $[a, b)$. If $f^{\Delta}(t) = 0$ on $[a, b)$, then f is a constant function.*

Proof.: *For any two points $t_1, t_2 \in [a, b)$, from the Mean Value Theorem 2.1.17 on $[t_1, t_2]$ we have*

$$(t_2 - t_1)f^{\Delta}(\xi_1) \le f(t_2) - f(t_1) \le (t_2 - t_1)f^{\Delta}(\xi_2)$$

$$0 \le f(t_2) - f(t_1) \le 0,$$

that is, $f(t_2) = f(t_1)$ for any $t_1, t_2 \in [a, b]$.

Corollary 2.1.19 *[26] Let f be continuous on $[a, b]$ and has a delta derivative at each point of $[a, b)$. Then,*

(1) If $f^{\Delta}(t) > 0$ on $[a, b)$, f is increasing.

(2) If $f^{\Delta}(t) \ge 0$ on $[a, b)$, f is nondecreasing.

(3) If $f^{\Delta}(t) < 0$ on $[a, b)$, f is decreasing.

(4) If $f^{\Delta}(t) \le 0$ on $[a, b)$, f is nondecreasing.

2.2 Integration on time scales

In this section we introduce the concept of integration on time scales, and define definite (Cauchy) integrals and indefinite integrals.

Definition 2.2.1 *[16] A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated if its right-limits exist at all right-dense points of $\mathbb{T}$ and its left limits exist at left-dense points.*

Example 2.2.2 *[26] Let $\mathbb{T} = \mathbb{N} = \{1, 2, 3, ...\}$ and define*

$$f(t) = \frac{t^2}{t-1}, \; g(t) = \frac{t}{t+1}, t \in \mathbb{T}.$$

Since 1 is left-dense, we compute

$$\lim_{t\to 1^-} \frac{t^2}{t-1} = -\infty, \textit{and} \; \lim_{t\to 1^-} \frac{t}{t+1} = \frac{1}{2}.$$

Then $f(t)$ is not regulated and $g(t)$ is regulated.

Definition 2.2.3 *[16] A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called pre-differentiable with region of differentiation D if:*

(1) $D \subset \mathbb{T}^k$,

(2) $\mathbb{T}^k \backslash D$ is countable and contains no right-scattered elements of $\mathbb{T}$,

(3) f is differentiable at each $t \in D$.

Definition 2.2.4 *[16] A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at right-dense points of $\mathbb{T}$ and its left limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions is denoted as*

$$C_{rd} = C_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R}).$$

The following theorem gives the relation between continuous, regulated and rd-continuous functions.

Theorem 2.2.5 *[16] Let $f : \mathbb{T} \to \mathbb{R}$.*

(1) If f is continuous, then f is rd-continuous.

(2) If f is rd-continuous, then it is regulated.

(3) The forward jump operator $\sigma(t)$ is rd-continuous.

(4) If f is regulated or rd-countinuous the $f^{\sigma}(t) = f(\sigma(t))$ is also regulated or rd-continuous.

(5) If f is continuous and $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then so is $f \circ g$.

Now, we define the integral on time scales.

Definition 2.2.6 *[16] Let $f : \mathbb{T} \to \mathbb{R}$ be a regulated function. Then there exists a function F which is pre-differentiable with region D such that*

$$F^{\Delta}(t) = f(t) \ \ for\ all \ \ t \in D.$$

Such a function F is called pre-antiderivative of f.
The indefinite integral of a regulated function f is defined as

$$\int f(t)\Delta t = F(t) + C,$$

where C is arbirary constant.
The Cauchy integral of f is defined as

$$\int_r^s f(t)\Delta t = F(s) - F(r), \quad r, s \in \mathbb{T}.$$

If

$$F^{\Delta}(t) = f(t) \ \ for\ all \ \ t \in \mathbb{T}^k.$$

then F is called antiderivative of f.

Theorem 2.2.7 *[16] Every rd-continuous function has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) = \int_{t_0}^t f(\tau)\Delta\tau \ \ for \ \ t \in \mathbb{T},$$

is an antiderivative of f.

The next theorem gives details of the computation of Cauchy integrals on time scales for the cases of dense and isolated points.

Theorem 2.2.8 *[16] Let $a, b \in \mathbb{T}, f \in C_{rd}$.*

(1) If $\mathbb{T} = \mathbb{R}$, *then*

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt.$$

(2) If $[a, b]$ *has only isolated points,then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t) & \text{if} \quad a < b \\ 0 & \text{if} \quad a = b \\ -\sum_{t\in[b,a)} \mu(t)f(t) & \text{if} \quad a > b \end{cases}.$$

(3) In particular, if $\mathbb{T} = h\mathbb{Z} = \{......, -2h, -h, 0, h, 2h, 3h, ...\}$, *then*

$$\int_a^b f(t)\Delta t = \begin{cases} -\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h & \text{if} \quad a < b \\ 0 & \text{if} \quad a = b \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} f(kh)h & \text{if} \quad a < b \end{cases}.$$

We now present some examples of integration on various time scales.

Example 2.2.9 *[26] Let* $\mathbb{T} = \mathbb{Z}, f(t) = 3t^2 + 5t + 2$ *and* $g(t) = t^3 + t^2$. *Then we have*

$$\begin{aligned} g^{\Delta}(t) &= (t^3)^{\Delta} + (t^2)^{\Delta} \\ &= \sigma^2(t) + t\sigma(t) + t^2 + \sigma(t) + t \\ &= (t+1)^2 + t(t+1) + t^2 + t + 1 + t \\ &= 3t^2 + 5t + 2. \end{aligned}$$

Hence,

$$\int (3t^2 + 5t + 2)\Delta t = \int (g(t))^{\Delta}\Delta t = g(t) + C = t^3 + t^2 + C.$$

Example 2.2.10 *[26] Let* $\mathbb{T} = 2^{\mathbb{N}}, f(t) = \frac{2}{t}\sin(\frac{t}{2})\cos(\frac{3t}{2})$ *and* $g(t) = \sin t$. *Then*

$$\begin{aligned} g^{\Delta}(t) &= \frac{\sin(\sigma(t)) - \sin t}{\sigma(t) - t} \\ &= \frac{\sin(2t) - \sin t}{2t - t} \\ &= \frac{2\sin\frac{t}{2}\cos\frac{3t}{2}}{t} \\ &= f(t). \end{aligned}$$

Hence,

$$\int f(t)\Delta t = \int (g(t))^{\Delta}\Delta t = \int (\sin t)^{\Delta}\Delta t = \sin t + C.$$

Example 2.2.11 *[26] Let* $\mathbb{T} = \mathbb{N}_0^2, f(t) = \dfrac{1}{1+2\sqrt{t}}\log\dfrac{(\sqrt{t}+1)^2}{t}$ *and* $g(t) = \log t$. *Then,*

$$\begin{aligned} g^{\Delta}(t) &= \frac{\log(\sigma(t)) - \log t}{\sigma(t) - t} \\ &= \frac{\log(\sqrt{t}+1)^2 - \log t}{(\sqrt{t}+1)^2 - t} \\ &= \frac{1}{t+2\sqrt{t}+1-t}\log\frac{(\sqrt{t}+1)^2}{t} \\ &= \frac{1}{1+2\sqrt{t}}\log\frac{(\sqrt{t}+1)^2}{t}. \end{aligned}$$

We conclude that

$$\int f(t)\Delta t = \int (g(t))^{\Delta}\Delta t = \int \frac{1}{1+2\sqrt{t}}\log\frac{(\sqrt{t}+1)^2}{t}\Delta t = \log t + C.$$

Theorem 2.2.12 *[26] If* $f \in C_{rd}(\mathbb{T})$ *and* $t \in \mathbb{T}^k$, *then*

$$F(t) = \int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

Proof.

Since $f \in C_{rd}(\mathbb{T})$, *then from Theorem 2.2.7,* f *has antiderivative* F *such that* $F^{\Delta} = f$ *and*

$$F(t) = \int_{t_0}^{t} f(\tau)\Delta\tau.$$

Then

$$\begin{aligned} \int_t^{\sigma(t)} f(\tau)\Delta\tau &= F(\sigma(t)) - F(t) \\ &= \frac{F(\sigma(t)) - F(t)}{\sigma(t) - t}(\sigma(t) - t) \\ &= (F^{\Delta}(t))\mu(t) \\ &= f(t)\mu(t). \end{aligned}$$

At the end of this section we give the properties of Cauchy integral on time scales which are similar to the properties of definite integrals on the set of real numbers.

Theorem 2.2.13 *[16] If* $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$ *and* $f, g \in C_{rd}(\mathbb{T})$, *then*

(1) $\int_a^b (f(t) + g(t))\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$

(2) $\int_a^b \alpha f(t)\Delta t = \alpha \int_a^b f(t)\Delta t$

(3) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$

(4) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$

(5) $\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\Delta t$

(5) $\int_a^b f(t)g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t$

(7) $\int_a^a f(t)\Delta t = 0$

(8) *If* $|f(t)| \leq g(t)$ *on* $[a, b]$,*then*

$$\left| \int_a^b f(t)\Delta t \right| \leq \int_a^b g(t)\Delta t$$

(9) *If* $f(t) \geq 0$ *for all* $t \in [a, b]$, *then* $\int_a^b f(t)\Delta t \geq 0.$

2.3 Elementary functions on time scales

In this section we introduce some elementary functions on time scales such as the exponential function and trigonometric functions. In the definition of the exponential function some preliminaries from complex calculus will be needed. For detailed information one can read the related chapters in [16, 17, 26] and [13].

2.3.1 Exponential function on time scales and its properties

Before the definition of exponential function on time scales we recall some notions of complex numbers and functions on complex numbers.

Definition 2.3.1 *[26] Let* $h > 0$.

(1) The Hilger complex numbers are defined and denoted by

$$\mathbb{C}_h = \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}.$$

(2) *The set* $\mathbb{Z}_h$ *is defined as the strip*

$$\mathbb{Z}_h = \left\{ z \in \mathbb{C} \mid -\frac{\pi}{h} < Im(z) \leq \frac{\pi}{h} \right\}.$$

For $h = 0$*, we set* $\mathbb{C}_0 = \mathbb{Z}_0 = \mathbb{C}$.

Definition 2.3.2 *[26] For* $h > 0$*, we define the cylinder transformation* $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ *by*

$$\xi_h(z) = \frac{1}{h} Log(1 + zh),$$

where Log is the principal branch of the logarithm function on $\mathbb{C}$*. Moreover, we define* $\xi_0(z) = z$ *for all* $z \in \mathbb{C}$.

Remark 2.3.3 *[26] We note that*

$$\xi_h^{-1}(z) = \frac{e^{zh} - 1}{h}$$

for $z \in \mathbb{Z}_h$.

Definition 2.3.4 *[13] The circle plus addition* $\oplus$ *on* $\mathbb{C}_h$ *is defined by*

$$z \oplus w = z + w + zwh.$$

Theorem 2.3.5 *[16]* $(\mathbb{C}_h, \oplus)$ *is an Abelian group.*

Proof.

Let $z, w \in \mathbb{C}_h$*, that is,* $z, w \neq -\frac{1}{h}$.

(1) $\mathbb{C}_h$ *is closed with respect to* $\oplus$*. We will show that* $z \oplus w$ *is in* $\mathbb{C}_h$*. Since*

$$\begin{aligned} h(z \oplus w) + 1 &= h(z + w + zwh) + 1 \\ &= 1 + hz + hw + zwh^2 \\ &= 1 + hz + hw(1 + hz) \\ &= (1 + hw)(1 + hz) \\ &\neq 0, \end{aligned}$$

we conclude that $z \oplus w \in \mathbb{C}_h$.

(2) $\mathbb{C}_h$ has a zero element with respect to $\oplus$ which is 0.

First, note that $0 \in \mathbb{C}_h$ and for any $z \in \mathbb{C}$

$$z \oplus 0 = z + 0 + z.0.h = z.$$

(3) Every $z \in \mathbb{C}_h$ has an inverse $z^{-1} \in \mathbb{C}_h$.

We claim that $z^{-1} = -\dfrac{z}{1+zh}$ is an inverse of z. Clearly,

$$\begin{aligned} z \oplus z^{-1} &= z + \left(-\frac{z}{1+zh}\right) + z\left(-\frac{z}{1+zh}\right).h \\ &= \frac{z(1+zh) - z - z^2h}{1+zh} \\ &= 0. \end{aligned}$$

Note that

$$1 + hz^{-1} = 1 + h.\left(-\frac{z}{1+zh}\right) = \frac{1+hz-hz}{1+hz} = \frac{1}{1+hz} \neq 0.$$

Therefore, $z^{-1} \in \mathbb{C}_h$.

(4) $(\mathbb{C}_h, \oplus)$ is associative.

$$\begin{aligned} (z \oplus w) \oplus v &= (z + w + zwh) \oplus v \\ &= z + w + zwh + v + (z + w + zwh)vh \\ &= z + w + zwh + v + zvh + wvh + zwvh^2, \end{aligned}$$

and

$$\begin{aligned} z \oplus (w \oplus v) &= z + (w \oplus v) + z(w \oplus v)h \\ &= z + w + v + wvh + z(w + v + wvh)h \\ &= z + w + v + wvh + zwh + zvh + zwvh^2. \end{aligned}$$

Consequently,

$$z \oplus (w \oplus v) = (z \oplus w) \oplus v.$$

(5) $(\mathbb{C}_h, \oplus)$ is Abelian(commutative).

$$z \oplus w = z + w + zwh = w + z + wzh = w \oplus z.$$

Definition 2.3.6 *[26] Let $z \in \mathbb{C}_h$. The circle minus $\ominus$ for $z \in \mathbb{C}$ is defined as*

$$\ominus z = \frac{-z}{1+zh}.$$

Remark 2.3.7 *[26] Let $z, w \in \mathbb{C}_h$. Notice that circle minus $\ominus$ subtraction can be writen as*

$$z \ominus w = z \oplus (\ominus w).$$

Observe that we have

$$\begin{aligned} z \ominus w &= z \oplus (\ominus w) \\ &= z + (\ominus w) = z(\ominus w)h \\ &= z - \frac{w}{1+wh} - \frac{zwh}{1+wh} \\ &= \frac{z + zwh - w - zwh}{1+wh} \\ &= \frac{z-w}{1+wh}. \end{aligned}$$

Then

$$z \ominus w = \frac{z-w}{1+hw}.$$

Definition 2.3.8 *[16] A function $f : \mathbb{T} \to \mathbb{R}$ is called regressive if*

$$1 + \mu(t)f(t) \neq 0 \quad for\ all\ \ t \in \mathbb{T}^k$$

holds. The set of all regressive and rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ or $\mathcal{R}(\mathbb{T})$ or $\mathcal{R}(\mathbb{T}, \mathbb{R})$.

Definition 2.3.9 *[16] In $\mathcal{R}$, the circle plus addition is defined by*

$$f \oplus g = f + g + \mu f g, \quad f, g \in \mathcal{R},$$

and for $f \in \mathcal{R}$, the circle minus is defined as

$$\ominus f = -\frac{f}{1+\mu f},$$

and the circle minus subtraction $\ominus$ on $\mathcal{R}$ is defined as

$$f \ominus g = f \oplus (\ominus g), \quad f, g \in \mathcal{R}.$$

Theorem 2.3.10 *[16] Let $f, g \in \mathcal{R}$. Then:*

(1) $f \ominus f = 0$,

(2) $\ominus(\ominus f) = f$,

(3) $f \ominus g \in \mathcal{R}$,

(4) $\ominus(f \ominus g) = g \ominus f$,

(5) $\ominus(f \oplus g) = (\ominus f) \oplus (\ominus g)$,

(6) $f \oplus \dfrac{g}{1+\mu f} = f + g$.

Definition 2.3.11 *[26] For $f \in \mathcal{R}$, the generalized exponential function is defined by*

$$e_f(t,s) = e^{\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} = e^{\int_s^t \frac{1}{\mu(\tau)} Log(1+\mu(\tau)f(\tau))\Delta\tau}, \quad for\ s,t \in \mathbb{T}.$$

Some properties of the exponential function $e_f(t,s)$ are listed below.
Let $f \in \mathcal{R}$. Then we have

(1) $e_f(t,r)e_f(r,s) = e_f(t,s)$ for all $t,r,s \in \mathbb{T}$,

(2) $e_0(t,s) = 1$ and $e_f(t,t) = 1$,

(3) $e_f^{\Delta}(t,t_0) = f(t).e_f(t,t_0)$,

(4) $e_f(\cdot,t_0)$, is a solution of the Cauchy problem

$$y^{\Delta}(t) = f(t)y(t), \quad y(t_0) = 1,$$

(5) $e_f(\sigma(t),s) = (1+\mu(t)f(t)).e_f(t,s)$,

(6) $e_f(t,s) = \dfrac{1}{e_f(s,t)} = e_{\ominus f}(s,t)$,

(7) $e_f(t,s).e_g(t,s) = e_{f\oplus g}(t,s)$,

(8) $\dfrac{e_f(t,s)}{e_g(t,s)} = e_{f\ominus g}(t,s)$,

(9) $e_f(t,\sigma(s)).e_f(s,r) = \dfrac{1}{1+\mu(s)f(s)}.e_f(t,r)$,

(10) $e^{\Delta}_{f\ominus g}(t,t_0) = \dfrac{(f(t)-g(t)).e_f(t,t_0)}{e_g(\sigma(t),t_0)}$,

(11) $(e_f(c,t))^{\Delta} = -f(t)e_f(c,\sigma(t))$,

(12) $\displaystyle\int_a^b f(t)e_f(c,\sigma(t))\Delta t = e_f(c,a) - e_f(c,b)$.

Next we give some examples of exponential function.

Example 2.3.12 *Let* $\alpha : \mathbb{T} \to \mathbb{R}$ *be regulated, that is,* $1 + \alpha(t)\mu(t) \neq 0$ *for all* $t \in \mathbb{T}$. *Let also,* $t_0, t \in \mathbb{T}, t_0 < t$.

(1) Let $\mathbb{T} = h\mathbb{Z}, h > 0$. *Every point in* $\mathbb{T}$ *is isolated and* $\mu(t) = h$ *for every* $t \in \mathbb{T}$. *Then*

$$\begin{aligned} e_\alpha(t, t_0) &= e^{\int_{t_0}^{t} \frac{1}{\mu(\tau)} Log(1+\alpha(\tau)\mu(\tau))\Delta\tau} \\ &= e^{\sum\limits_{s\in[t_0,t)} \frac{1}{\mu(s)} Log(1+\alpha(s)\mu(s))\mu(s)} \\ &= e^{\sum\limits_{s\in[t_0,t)} Log(1+h\alpha(s))} \\ &= \prod_{s\in[t_0,t)} (1+h\alpha(s)). \end{aligned}$$

If α *is a constant, then*

$$\begin{aligned} e_\alpha(t, t_0) &= \prod_{s\in[t_0,t)} (1+h\alpha) \\ &= (1+h\alpha)^{t-t_0}. \end{aligned}$$

(2) Let $\mathbb{T} = q^{\mathbb{N}_0}, q > 1$. *Every point of* $\mathbb{T}$ *is isolated and* $\mu(t) = (q-1)t$ *for all* $t \in \mathbb{T}$. *Then*

$$\begin{aligned} e_\alpha(t, t_0) &= e^{\int_{t_0}^{t} \frac{1}{\mu(\tau)} Log(1+\alpha(\tau)\mu(\tau))\Delta\tau} \\ &= e^{\sum\limits_{s\in[t_0,t)} \frac{1}{\mu(s)} Log(1+\alpha(s)\mu(s))\mu(s)} \\ &= e^{\sum\limits_{s\in[t_0,t)} Log(1+\alpha(s)\mu(s))} \\ &= e^{\sum\limits_{s\in[t_0,t)} Log(1+(q-1)s\alpha(s))} \\ &= \prod_{s\in[t_0,t)} (1+(q-1)s\alpha(s)). \end{aligned}$$

(3) $\mathbb{T} = \mathbb{N}_0^k, k \in \mathbb{N}$. *Every point of* $\mathbb{T}$ *is isolated and*

$$\mu(t) = \left(\sqrt[k]{t} + 1\right)^k$$

Then

$$\begin{aligned}
e_\alpha(t,t_0) &= e^{\displaystyle\int_{t_0}^{t} \frac{1}{\mu(\tau)} Log(1+\alpha(\tau)\mu(\tau))\Delta\tau} \\
&= e^{\displaystyle\sum_{s\in[t_0,t)} \frac{1}{\mu(s)} Log(1+\alpha(s)\mu(s))\mu(s)} \\
&= e^{\displaystyle\sum_{s\in[t_0,t)} Log(1+\alpha(s)\mu(s))} \\
&= \prod_{s\in[t_0,t)} (1+\alpha(s)\mu(s)) \\
&= \prod_{s\in[t_0,t)} \left(1+\left(\left(\sqrt[k]{s}+1\right)^k - s\right)\alpha(s)\right).
\end{aligned}$$

2.3.2 Hyperbolic and trigonometric functions

Definition 2.3.13 *[26] (Hyperbolic Functions) If $f \in \mathcal{R}$ and $-\mu f^2 \in \mathcal{R}$, then we define the hyperbolic functions $\cosh_f$ and $\sinh_f$ by*

$$\cosh_f(t,s) = \frac{e_f(t,s)+e_{-f}(t,s)}{2} \quad \text{and} \quad \sinh_f(t,s) = \frac{e_f(t,s)-e_{-f}(t,s)}{2}.$$

Theorem 2.3.14 *[26] Let $f \in \mathcal{R}$ and $-\mu f^2 \in \mathcal{R}$, then we have*

(1) $\cosh_f^\Delta(t,s) = f\sinh_f(t,s)$,

(2) $\sinh_f^\Delta(t,s) = f\cosh_f(t,s)$,

(3) $\cosh_f^2(t,s) - \sinh_f^2(t,s) = e_{-\mu f^2}(t,s)$.

Proof.

(1)

$$\begin{aligned}
\cosh_f^\Delta(t,s) &= \left(\frac{e_f(t,s)+e_{-f}(t,s)}{2}\right)^\Delta \\
&= \frac{fe_f(t,s)-fe_{-f}(t,s)}{2} \\
&= f\sinh_f(t,s).
\end{aligned}$$

(2)

$$\begin{aligned}\sinh_f^{\Delta}(t,s) &= \left(\frac{e_f(t,s)-e_{-f}(t,s)}{2}\right)^{\Delta}\\ &= \frac{fe_f(t,s)+fe_{-f}(t,s)}{2}\\ &= f\cosh_f(t,s).\end{aligned}$$

(3)

$$\begin{aligned}\cosh_f^2(t,s)-\sinh_f^2(t,s) &= \left(\frac{e_f(t,s)+e_{-f}(t,s)}{2}\right)^2-\left(\frac{e_f(t,s)-e_{-f}(t,s)}{2}\right)^2\\ &= \frac{e_f^2(t,s)+2e_f(t,s)e_{-f}(t,s)+e_{-f}^2(t,s)}{4}\\ &- \frac{e_f^2(t,s)-2e_f(t,s)e_{-f}(t,s)+e_{-f}^2(t,s)}{4}\\ &= e_f(t,s)e_{-f}(t,s)\\ &= e_{f\oplus(-f)}(t,s)\\ &= e_{-\mu f^2}(t,s).\end{aligned}$$

Definition 2.3.15 *[26] (Trigonometric Functions) If $f\in\mathcal{R}$ and $\mu f^2\in\mathcal{R}$, then we define the trigonometric functions $\cos_f$ and $\sin_f$ by*

$$\cos_f(t,s)=\frac{e_{if}(t,s)+e_{-if}(t,s)}{2}\quad \textit{and}\quad \sin_f(t,s)=\frac{e_{if}(t,s)-e_{-if}(t,s)}{2i}.$$

Theorem 2.3.16 *[26] Let $f\in\mathcal{R}$ and $-\mu f^2\in\mathcal{R}$. Then*

(1) $\cos_f^{\Delta}(t,s)=-f\sin_f(t,s)$,

(2) $\sin_f^{\Delta}(t,s)=f\cos_f(t,s)$,

(3) $\cos_f^2(t,s)+\sin_f^2(t,s)=e_{\mu f^2}(t,s)$.

Proof.

(1)

$$\begin{aligned}\cos_f^{\Delta}(t,s) &= \left(\frac{e_{if}(t,s)+e_{-if}(t,s)}{2}\right)^{\Delta}\\ &= \frac{ife_{if}(t,s)-ife_{-if}(t,s)}{2}\\ &= -f\frac{e_{if}(t,s)-e_{-if}(t,s)}{2i}\\ &= -f\sin_f(t,s).\end{aligned}$$

(2)

$$\begin{aligned}\sin_f^{\Delta}(t,s) &= \left(\frac{e_{if}(t,s)-e_{-if}(t,s)}{2i}\right)^{\Delta}\\ &= \frac{ife_{if}(t,s)+ife_{-if}(t,s)}{2i}\\ &= f\cos_f(t,s).\end{aligned}$$

(3)

$$\begin{aligned}\cos_f^2(t,s)+\sin_f^2(t,s) &= \left(\frac{e_{if}(t,s)+e_{-if}(t,s)}{2}\right)^2+\left(\frac{e_{if}(t,s)-e_{-if}(t,s)}{2i}\right)^2\\ &= \frac{e_{if}^2(t,s)+2e_{if}(t,s)e_{-if}(t,s)+e_{-if}^2(t,s)}{4}\\ &- \frac{e_{if}^2(t,s)-2e_{if}(t,s)e_{-if}(t,s)+e_{-if}^2(t,s)}{4}\\ &= e_{if}(t,s)e_{-if}(t,s)\\ &= e_{if\oplus(-if)}(t,s)\\ &= e_{\mu f^2}(t,s).\end{aligned}$$

2.3.3 Monomials on time scales

Finally we define the monomials on time scales. Let $s, t \in \mathbb{T}$. Define the polynomials $g_n(t,s)$ and $h_n(t,s)$ as follows [26].

$$g_0(t,s) = h_0(t,s) = 1,$$

$g_{k+1}(t,s) = \int_s^t g_k(\sigma(\tau),s)\Delta\tau,$
$h_{k+1}(\tau,s) = \int_s^t h_k(\tau,s)\Delta\tau$, for $k = 0,1,2,.......$
We compute $g_1(t,s), h_1(t,s), g_2(t,s)$ and $h_2(t,s)$ as follows.

$$g_1(t,s) = \int_s^t g_0(\sigma(\tau),s)\Delta\tau = \int_s^t \Delta\tau = t-s,$$

$$g_2(t,s) = \int_s^t g_1(\sigma(\tau),s)\Delta\tau = \int_s^t (\sigma(\tau)-s)\Delta\tau,$$

$$h_1(t,s) = \int_s^t h_0(\tau,s)\Delta\tau = \int_s^t \Delta\tau = t-s,$$

$$h_2(t,s) = \int_s^t h_1(\tau,s)\Delta\tau = \int_s^t (\tau-s)\Delta\tau.$$

Note that

$$g_k^{\Delta}(t,s) = g_{k-1}(\sigma(t),s), \quad h_k^{\Delta}(t,s) = h_{k-1}(t,s), \quad k \in \mathbb{N}.$$

Example 2.3.17 *Let* $\mathbb{T} = 2^{\mathbb{N}_0}$*.Then* $\sigma(t) = 2t, \mu(t) = \sigma(t) - t = t$.
We have

$$h_0(s,t) = 1, \quad g_0(s,t) = 1,$$

$$h_1(s,t) = t - s, \quad g_1(s,t) = t - s, \quad h_2(s,t) = \int_s^t (\tau - s)\Delta\tau.$$

Note that $\left(\frac{x^2}{3}\right)^\Delta = \frac{1}{3}(\sigma(x) + x) = \frac{3x}{3} = x$.

$$\begin{aligned} h_2(s,t) &= \int_s^t \left\{ \left(\frac{\tau^2}{3}\right)^\Delta - (s\tau)^\Delta \right\} \Delta\tau = \left(\frac{\tau^2}{3} - s\tau\right)\Big|_s^t \\ &= \left(\frac{t^2}{3} - st\right) - \left(\frac{s^2}{3} - s^2\right) = \frac{t^2 - s^2}{3} - s(t-s) \\ &= (t-s)\left(\frac{t+s}{3} - s\right) \\ &= \frac{(t-s)(t-2s)}{3}, \end{aligned}$$

and

$$g_2(s,t) = \int_s^t (\sigma(\tau) - s)\Delta\tau = \int_s^t (2\tau - s)\Delta\tau.$$

Note that $\left(\frac{2x^2}{3}\right)^\Delta = \frac{2}{3}(\sigma(x) + x) = 2x$*, so that,*

$$\begin{aligned} g_2(s,t) &= \int_s^t \left\{ \left(\frac{2\tau^2}{3}\right)^\Delta - (s\tau)^\Delta \right\} \Delta\tau = \left(\frac{2\tau^2}{3} - s\tau\right)\Big|_s^t \\ &= \left(\frac{2t^2}{3} - st\right) - \left(\frac{2s^2}{3} - s^2\right) \\ &= \frac{(2t-s)(t-s)}{3}. \end{aligned}$$

Lemma 2.3.18 *[26] Let* $\mathbb{T}$ *be a time scales. If* f *is n-times differentiable and* $p_k : \mathbb{T} \to \mathbb{R}$ *are differentiable functions for* $k = 0, 1, ..., n-1$ *such that*

$$p_{k+1}^\Delta(t) = p_k^\sigma(t) = p_k(\sigma(t)), \quad k = 0, ..., n-1,$$

then

$$\left(\sum_{k=0}^{n-1} (-1)^k f^{\Delta^k}(t) p_k(t)\right)^\Delta = (-1)^{n-1} f^{\Delta^n}(t) p_{n-1}^\sigma(t) + f(t) p_0^\Delta(t).$$

Proof.:

$$
\begin{aligned}
\left(\sum_{k=0}^{n-1}(-1)^k f^{\Delta^k}(t)p_k(t)\right)^{\Delta} &= \sum_{k=0}^{n-1}(-1)^k\left(f^{\Delta^k}(t)p_k(t)\right)^{\Delta}\\
&= \sum_{k=0}^{n-1}(-1)^k\left(f^{\Delta^{k+1}}(t)p_k^{\sigma}(t)+f^{\Delta^k}(t)p_k^{\Delta}(t)\right)\\
&= \sum_{k=0}^{n-2}(-1)^k f^{\Delta^{k+1}}(t)p_k^{\sigma}(t)+(-1)^{n-1}f^{\Delta^n}(t)p_{n-1}^{\sigma}(t)\\
&+ (-1)^0 f(t)p_0^{\Delta}(t)+\sum_{k=1}^{n-1}(-1)^k f^{\Delta^k}(t)p_k^{\Delta}(t)\\
&= \sum_{l=0}^{n-2}(-1)^l f^{\Delta^{l+1}}(t)p_l^{\sigma}(t)+\sum_{l=0}^{n-2}(-1)^{l+1}f^{\Delta^{l+1}}(t)p_{l+1}^{\Delta}(t)\\
&+ (-1)^{n-1}f^{\Delta^n}(t)p_{n-1}^{\sigma}(t)+f(t)p_0^{\Delta}(t)\\
&= \sum_{l=0}^{n-2}(-1)^l f^{\Delta^{l+1}}(t)\left(p_l^{\sigma}(t)-p_{l+1}^{\Delta}(t)\right)+f(t)p_0^{\Delta(t)}+(-1)^{n-1}f^{\Delta^n}(t)p_{n-1}^{\sigma}(t)\\
&= f(t)p_0^{\Delta}(t)+(-1)^{n-1}f^{\Delta^n}(t)p_{n-1}^{\sigma}(t).
\end{aligned}
$$

Definition 2.3.19 *[1] (Taylor series on $\mathbb{T}$)*
Let f be infinitely many times differentiable at some $\alpha \in \mathbb{T}$, then the Taylor series of f about α is defined as

$$f(t)=\sum_{n=0}^{\infty} f^{\Delta^n}(\alpha)h_n(t,\alpha)$$

and converges to f on some interval containing α.

Theorem 2.3.20 *[1] (Taylor's Formula)*
Let $n \in \mathbb{N}$. Suppose that f is n-times differentiable on $\mathbb{T}^{k^n}$. Let also, $\alpha \in \mathbb{T}^{k^{n-1}}, t \in \mathbb{T}$. Then

$$f(t)=\sum_{k=0}^{n-1}h_k(t,\alpha)f^{\Delta^k}(\alpha)+\int_{\alpha}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau,$$

where the integral on the right-hand-side is the remainder term.

We now compute the Taylor series of some elementary functions on a general time scales $\mathbb{T}$.

Example 2.3.21 *Compute the Taylor series of $e_c(t,\alpha)$ where c is a constant.*

We have $f(t) = e_c(t,\alpha)$, *with c constant. Then*

$$\begin{aligned}
&f^{\Delta}(t) = (e_c(t,\alpha))^{\Delta} = c.e_c(t,\alpha), && f^{\Delta}(\alpha) = c\\
&f^{\Delta^2}(t) = c^2.e_c(t,\alpha), && f^{\Delta^2}(\alpha) = c^2\\
&f^{\Delta^3}(t) = c^3.e_c(t,\alpha), && f^{\Delta^3}(\alpha) = c^3\\
&\vdots\\
&f^{\Delta^n}(t) = c^n.e_c(t,\alpha), && f^{\Delta^n}(\alpha) = c^n.
\end{aligned}$$

We conclude that the Taylor series of $e_c(t,\alpha)$ *about* α *is*

$$\sum_{n=0}^{\infty} c^n h_n(t,\alpha) = h_0(t,\alpha) + ch_1(t,\alpha) + \ldots + c^n h_n(t,\alpha) + \ldots$$

Example 2.3.22 *Compute the Taylor series of* $\cos_c(t,\alpha)$ *and* $\sin_c(t,\alpha)$ *for an arbitrary constant c and* $\alpha \in \mathbb{T}$.

(1)

$$\begin{aligned}
&f(t) = \cos_c(t,\alpha), && f(\alpha) = \cos_c(\alpha,\alpha) = 1 \;\; \textit{where} \;\; c \;\; \textit{constant}.\\
&f^{\Delta}(t) = -c\sin_c(t,\alpha), && f^{\Delta}(\alpha) = -c\sin_c(\alpha,\alpha) = 0.\\
&f^{\Delta^2}(t) = -c^2\cos_c(t,\alpha), && f^{\Delta^2}(\alpha) = -c^2.\\
&f^{\Delta^3}(t) = c^3\sin_c(t,\alpha), && f^{\Delta^3}(\alpha) = 0.\\
&f^{\Delta^4}(t) = c^4\cos_c(t,\alpha), && f^{\Delta^4}(\alpha) = c^4.\\
&\vdots
\end{aligned}$$

$$f^{\Delta^n}(\alpha) = \begin{cases} 0, & \textit{if} \;\; n = 2k+1 \\ (-1)^k c^{2k}, & \textit{if} \;\; n = 2k \end{cases} .$$

Then, the Taylor series of $\cos_c(t,\alpha)$ *is*

$$\begin{aligned}
\sum_{k=0}^{\infty} f^{\Delta^{2k}}(\alpha) h_{2k}(t,\alpha) &= \sum_{k=0}^{\infty} (-1)^k.c^{2k}.h_{2k}(t,\alpha)\\
&= h_0(t,\alpha) - c^2 h_2(t,\alpha) + c^4 h_4(t,\alpha) + \ldots
\end{aligned}$$

(2)

$$
\begin{aligned}
&f(t) = \sin_c(t,\alpha), && f(\alpha) = \sin_c(\alpha,\alpha) = 0 \ \ \textit{where} \ \ c \ \ \textit{constant}.\\
&f^{\Delta}(t) = c\cos_c(t,\alpha), && f^{\Delta}(\alpha) = c.\\
&f^{\Delta^2}(t) = -c^2\sin_c(t,\alpha), && f^{\Delta^2}(\alpha) = 0.\\
&f^{\Delta^3}(t) = -c^3\cos_c(t,\alpha), && f^{\Delta^3}(\alpha) = -c^3.\\
&f^{\Delta^4}(t) = c^4\sin_c(t,\alpha), && f^{\Delta^4}(\alpha) = 0.\\
&\vdots
\end{aligned}
$$

$$
f^{\Delta^n}(\alpha) = \begin{cases} 0, & \textit{if} \quad n = 2k \\ (-1)^k c^{2k+1}, & \textit{if} \quad n = 2k+1 \end{cases}.
$$

This yields the Taylor series of $\sin_c(t,\alpha)$ *as*

$$
\begin{aligned}
\sum_{k=0}^{\infty} f^{\Delta^{2k+1}}(\alpha) h_{2k+1}(t,\alpha) &= \sum_{k=0}^{\infty} (-1)^k c^{2k+1} h_{2k+1}(t,\alpha)\\
&= c h_1(t,\alpha) - c^3 h_3(t,\alpha) + c^5 h_5(t,\alpha) - \dots
\end{aligned}
$$

CHAPTER 3

DYNAMIC EQUATION ON TIME SCALES

Many recent publications on time scales are related with dynamic equations. Dynamic equations like the time scales unify the continuous case which is the differential equations and the discrete case, that is, difference equations. The theory of linear dynamic equations and systems of linear dynamic equations have been stuied very extensively as can be seen from recent papers on the subject [9, 34, 2, 8, 15, 23, 25, 24]
Certain problems in biology, economy and other fields are represented by mathematical models which are both continuous and discrete. Therefore, the dynamic equations are the best option when dealing with this type of problems.

3.1 First order linear dynamic equations

In this section we will discuss the linear first order dynamic equations.

Definition 3.1.1 *[17] A linear first order dynamic equation is defined as*

$$y^{\Delta} = p(t)y + f(t), \tag{3.1}$$

where $p(t)$ and $f(t)$ are given functions. The equation

$$x^{\Delta} = -p(t)x^{\sigma} + f(t) \tag{3.2}$$

is called the adjoint equation.

The solutions of the equations (3.1) and (3.2) are given in the following theorems.

Theorem 3.1.2 *[17] Let $p : \mathbb{T} \to \mathbb{R}$ be rd-continuous and $1 + \mu(t)p(t) \neq 0$ for all*

$t \in \mathbb{T}$. Let also, $t_0 \in \mathbb{T}$ and $y_0 = y(t_0)$. Then the initial value problem

$$y^{\Delta} = p(t)y, \quad y(t_0) = y_0, \tag{3.3}$$

has a unique solution given by

$$y(t) = y_0 e_p(t, t_0). \tag{3.4}$$

Proof.

Let

$$y(t) = y_0 e_p(t, t_0).$$

Then we compute $y^{\Delta}(t)$ as

$$y^{\Delta}(t) = y_0(e_p(t, t_0))^{\Delta} = y_0 p(t) e_p(t, t_0) = p(t)y(t),$$

that is, y satisfies the dynamic equation. Also,

$$y(t_0) = y_0 e_p(t_0, t_0) = y_0,$$

that is, y also satisfies the initial condition. Then $y(t) = y_0 e_p(t, t_0)$ is a solution of the IVP.

To prove the uniqueness we assume that $y_1(t)$ and $y_2(t)$ are two solutions of the IVP. Then

$$y_1^{\Delta} = p(t)y_1, \quad y_2^{\Delta} = p(t)y_2,$$

and hence $(y_1 - y_2)^{\Delta} = p(t)(y_1 - y_2)$. On the other hand

$(y_1 - y_2)(t_0) = y_1(t_0) - y_2(t_0) = y_0 - y_0 = 0.$

Then the function

$$\psi(t) = y_1(t) - y_2(t)$$

satisfies

$$\psi^{\Delta}(t) = p(t)\psi, \ \psi(t_0) = 0.$$

Then we have $\psi(t) = 0 e_p(t, t_0) \equiv 0$ for all $t \in \mathbb{R}$, that is, $y_1(t) - y_2(t) \equiv 0$, which completes the proof of the uniqueness.

We discuss next the IVP for the nonhomogeneous adjoint equation (3.2).

Theorem 3.1.3 *[17] Consider the IVP*

$$y^{\Delta} = -p(t)y(\sigma(t)) + f(t),\ y(t_0) = y_0, \tag{3.5}$$

where $f, p : \mathbb{T} \to \mathbb{R}$ *are rd-continuous functions and* $1 + \mu(t)p(t) \neq 0$ *for all* $t \in \mathbb{T},\ y_0 \in \mathbb{R}$*. Then the solution of the IVP (3.5) is unique and is given as*

$$y(t) = e_{\ominus p}(t, t_0)y_0 + \int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\Delta\tau. \tag{3.6}$$

Proof.

Observe that

$$\begin{aligned} y^{\Delta}(t) &= \left(e_{\ominus p}(t, t_0)y_0 + \int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\Delta\tau\right)^{\Delta} \\ &= y_0 \ominus p(t)e_p(t, t_0) + \left(\int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\Delta\tau\right)^{\Delta}. \end{aligned}$$

The rule for differentiation under integral sign is given as

$$\left(\int_{t_0}^{t} f(t, u)\Delta u\right)^{\Delta} = f(\sigma(t), t) + \int_{t_0}^{t} f^{\Delta}(t, u)\Delta u.$$

Employing this rule we compute

$$\begin{aligned} y^{\Delta}(t) &= y_0(\ominus p(t))e_p(t, t_0) + e_{\ominus p}(\sigma(t), t)f(t) + \int_{t_0}^{t} e^{\Delta}_{\ominus p}(t, \tau)f(\sigma)\Delta\tau \\ &= y_0(\ominus p(t))e_p(t, t_0) + e_{\ominus p}(\sigma(t), t)f(t) + \int_{t_0}^{t} (\ominus p(t))e_{\ominus p}(t, \sigma)f(\sigma)\Delta\sigma. \end{aligned}$$

Since

$$\begin{aligned} e_{\ominus p}(\sigma(t), t)f(t) &= 1 + \mu(t)(\ominus p(t))e_{\ominus p}(t, t)f(t) \\ &= \frac{1}{1 + \mu(t)p(t)}f(t), \end{aligned}$$

we obtain

$$y^{\Delta}(t) = \frac{-p(t)}{1 + \mu(t)p(t)}y_0 e_{\ominus p}(t, t_0) + \frac{1}{1 + \mu(t)p(t)}f(t) - \frac{p(t)}{1 + \mu(t)p(t)}\int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\Delta\tau.$$

Upon multiplication by $1 + \mu(t)p(t)$ *we get*

$$\begin{aligned} (1 + \mu(t)p(t))y^{\Delta}(t) &= -p(t)e_{\ominus p}(t, t_0)y_0 + f(t) - p(t)\int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\Delta\tau \\ &= -p(t)\left(e_{\ominus p}(t, t_0)y_0 + \int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\Delta\tau\right) + f(t) \\ &= -p(t)y(t) + f(t). \end{aligned}$$

Hence,

$$\begin{aligned} y^{\Delta}(t) &= -\mu(t)p(t)y^{\Delta}(t) - p(t)y(t) + f(t) \\ &= -p(t)(\mu(t)y^{\Delta}(t) + y(t)) + f(t). \end{aligned}$$

Due to the fact that

$$y^{\Delta}(t) = \frac{y(\sigma(t)) - y(t)}{\mu(t)},$$

we get

$$y^{\Delta}(t) = -p(t)y(\sigma(t)) + f(t).$$

Hence, $y(t)$ is a solution of dynamic equation in (3.5). Also

$$y(t_0) = e_{\ominus p}(t_0, t_0)y_0 + \int_{t_0}^{t_0} e_{\ominus p}(t_0, \tau)f(\tau)\Delta\tau = y_0.$$

To show the uniqueness of the solution, let y_1, y_2 be two solutions of the IVP (3.5). Then we have

$$y_1^{\Delta} = -p(t)y_1(\sigma(t)) + f(t),$$

$$y_2^{\Delta} = -p(t)y_2(\sigma(t)) + f(t).$$

Therefore,

$$(y_1 - y_2)^{\Delta} = -p(t)(y_1(\sigma(t)) - y_2(\sigma(t))).$$

Define $\phi(t) = y_1(t) - y_2(t)$. Then $\phi(t)$ satisfies

$$\phi^{\Delta} = -p(t)\phi(t),$$

and moreover

$$\phi(t_0) = y_1(t_0) - y_2(t_0) = y_0 - y_0 = 0.$$

Then ϕ is the unique solution of

$$\phi^{\Delta} = -p(t)\phi(t), \ \phi(t_0) = 0. \tag{3.7}$$

As a result, we conclude that

$$\phi(t) = 0.e_{-p(t)}(t, t_0) \equiv 0.$$

and hence,

$$\phi(t) \equiv y_1(t) - y_2(t) \equiv 0,$$

which implies

$$y_1(t) = y_2(t).$$

As in the previous theorem we can proof the existence and uniqueness of solution of IVP for 1-st order nonhomogeneous dynamic equation (3.1).

Theorem 3.1.4 *[17] Consider the IVP*

$$y^{\Delta}(t) = p(t)y(t) + f(t), \; y(t_0) = y_0, \tag{3.8}$$

where $p, f : \mathbb{T} \to \mathbb{R}$ *are rd-continuous functions and* $1 + \mu(t)p(t) \neq 0, \; t_0 \in \mathbb{T}, y_0 \in \mathbb{R}$. *Then the IVP (3.8) has a unique solution in the form*

$$y(t) = e_p(t, t_0)y_0 + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Using the Existence-Uniquess Theorem 3.1.4 given above, we solve some examples of Initial Value Problems, associated with first order dynamic equations.

Example 3.1.5 *[17] Consider the IVP*

$$y^{\Delta} = 2y + 3^t, \;\; y(0) = 0,$$

where $\mathbb{T} = \mathbb{Z}$. *Here*

$$p(t) = 2, \;\; f(t) = 3^t, \;\; y_0 = 0,$$

and we know that

$$\sigma(t) = t + 1, \;\; \mu(t) = 1, \;\; t \in \mathbb{T}.$$

Then, the unique solution is given as

$$y(t) = \int_0^t e_2(t, \sigma(\tau))3^{\tau}\Delta\tau = \int_0^t e_2(t, \tau + 1)3^{\tau}\Delta\tau.$$

Empoying the Definitions 2.3.2 and 2.3.11 we obtain

$$e_2(t, \tau + 1) = e^{\int t^{\tau+1} \log 3\Delta\tau} = e^{(\log 3)(t-\tau-1)} = 3^{t-\tau-1}$$

Then

$$\begin{aligned} y(t) &= \int_0^t 3^{t-\tau-1}3^{\tau}\Delta\tau = \int_0^t 3^{t-1}\Delta\tau \\ &= 3^{t-1}\int_0^t \Delta\tau = t3^{t-1} \end{aligned}$$

is the unique solution of the IVP.

Example 3.1.6 *[17] Consider the IVP*

$$y^{\Delta} = 4y + t, \quad y(0) = 1,$$

on the time scale $\mathbb{T} = 2\mathbb{Z}$*. Here*

$$p(t) = 4, \quad f(t) = t, \quad y_0 = 1,$$

and

$$\sigma(t) = t + 2, \quad \mu(t) = 2, \quad t \in \mathbb{T}.$$

Then, by the Theorem 3.1.4, we obtain

$$y(t) = e_4(t, 0) + \int_0^t e_4(t, \sigma(\tau))\tau\Delta\tau = e_4(t, 0) + \int_0^t e_4(t, \tau + 2)\tau\Delta\tau$$

Using Definitions 2.3.2 and 2.3.11 of the exponential function we compute

$$e_4(t, 0) = 9^{\frac{t}{2}},$$

and

$$e_4(t, \tau + 2) = 9^{\frac{t-\tau-2}{2}}.$$

Then we obtain the unique solution of the IVP as

$$\begin{aligned} y(t) &= 3^t + \int_0^t 9^{\frac{t-\tau-2}{2}}\tau\Delta\tau \\ &= 3^t + 9^{\frac{t}{2}-1}\int_0^t 9^{-\frac{\tau}{2}}\tau\Delta\tau. \end{aligned}$$

3.2 Linear constant coefficient higher order homogeneous dynamic equations

In this section we introduce briefly the linear constant coefficient dynamic equation of higher order and some theoretical facts about their solutions.

Definition 3.2.1 *[17] Consider the linear n-th order constant coefficient dynamic equation*

$$\sum_{k=0}^{n} \alpha_k y^{\Delta^k} = 0. \tag{3.9}$$

The function

$$\varphi(\lambda) = \sum_{k=0}^{n} \alpha_k \lambda^k \tag{3.10}$$

is called the characteristic polynomial associated with (3.9).

Definition 3.2.2 *[17] The equation (3.9) is called regressive if for all zeros λ of φ, $\lambda \in \mathcal{R}$, that is,*

$$1 + \lambda\mu(t) \neq 0.$$

Theorem 3.2.3 *[17] Let $1 + \lambda\mu(t) \neq 0$ for all $t \in \mathbb{T}$. If λ is a zero of the characteristic polynomial, then $y(t) = e_\lambda(t, t_0)$ is a solution of the linear equation (3.9).*

Theorem 3.2.4 *[17] The equation (3.9) is regressive if and only if*

$$\sum_{k=0}^{n} \alpha_k(-\mu(t))^{n-k} \neq 0 \ \textit{ for all } \quad t \in \mathbb{T}.$$

Proof.: *Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$ be the zeros of φ counting multiplicities. Then*

$$\varphi(\lambda) = \alpha_n(\lambda - \lambda_1)(\lambda - \lambda_2).......(\lambda - \lambda_n).$$

Then we have the following equivalences:

$$\begin{array}{l} \lambda_v \in \mathcal{R} \textit{ for all } v = 1, 2, \ldots, n, \quad \Longleftrightarrow \\ 1 + \lambda_1\mu(t)) \neq 0, \quad 1 + \lambda_2\mu(t)) \neq 0, \quad , \ldots, 1 + \lambda_n\mu(t)) \neq 0, \quad \Longleftrightarrow \\ \alpha_n \prod_{v=1}^{n}(1 + \lambda_v\mu(t)) \neq 0. \end{array}$$

If $\mu(t) = 0$ then

$$0 \neq \alpha_n \prod_{v=1}^{n}(1 + \lambda_v\mu(t)) = \alpha_n.$$

If $\mu(t) \neq 0$ then

$$\begin{aligned} \alpha_n \textstyle\prod_{v=1}^{n}(1 + \lambda_v\mu(t)) &= \alpha_n(1 + \lambda_1\mu(t))(1 + \lambda_2\mu(t))(1 + \lambda_3\mu(t)) \ldots (1 + \lambda_n\mu(t)) \\ &= \alpha_n\mu(t)(\lambda_1 + \tfrac{1}{\mu(t)}).\mu(t)(\lambda_2 + \tfrac{1}{\mu(t)}).\mu(t)(\lambda_3 + \tfrac{1}{\mu(t)}) \ldots \mu(t)(\lambda_n \tfrac{1}{\mu(t)}) \\ &= \alpha_n\mu^n(t) \prod_{v=1}^{n}(\lambda_v + \frac{1}{\mu(t)}) = \alpha_n(-\mu(t))^n \prod_{v=1}^{n}(-\lambda_v + \frac{1}{-\mu(t)}) \\ &= (-\mu(t))^n\varphi(-\frac{1}{\mu(t)}) = (-\mu(t))^n \sum_{v=0}^{n} \alpha_k \left(-\frac{1}{\mu(t)}\right)^k \\ &= \sum_{v=0}^{n} \alpha_k(-\mu(t))^{n-k} \neq 0. \end{aligned}$$

This completes the proof.

Example 3.2.5 *[17] Suppose $\mathbb{T} = 2^{\mathbb{N}_0}$ and consider the equation*

$$y^{\Delta\Delta} - 3y^\Delta + 2y = 0.$$

We know that on $2^{\mathbb{N}_0}$ $\sigma(t) = 2t$, *and* $\mu(t) = t$. *The characteristic polynomial is*

$$\varphi(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0$$

and its roots are $\lambda = 2, \quad \lambda = 1$. *We can check whether the equation is regressive in two ways.*

First way : Since

$$1 + 2t \neq 0 \text{ for all } t \in \mathbb{T}$$

and

$$1 + t \neq 0 \text{ for all } t \in \mathbb{T},$$

then the equation is regressive by the definition.

Second way : Using Theorem 3.2.4, we have

$$1 - 3(-\mu(t)) + 2(-\mu(t))^2 = 2\mu^2(t) + 3\mu(t) + 1 \neq 0 \quad \text{for all} \quad t \in \mathbb{T},$$

since

$$2t^2 + 3t + 1 = 0, \text{ for } t_{1,2} = -1, -\frac{1}{2}.$$

Then the equation is regressive by the Theorem 3.2.4.

The two solutions of the equation are $y_1 = e_1(t, 1)$ *and* $y_2 = e_2(t, 1)$.

Lemma 3.2.6 *[17] Let* $\lambda \in \mathcal{R}$, *p be a differentiable function, and assume that* $\{P_k\}_{k\in\mathbb{N}_0}$ *is a sequence satisfying*

$$p_0 = p \text{ and } p_k^{\Delta} = \frac{p_{k+1}}{1 + \lambda\mu(t)} \text{ for } k \in \mathbb{N}_0.$$

Let $y(t) = p(t)e_\lambda(t, t_0)$. *Then*

$$y^{\Delta^k} = \left[\sum_{v=0}^{k} \begin{pmatrix} k \\ v \end{pmatrix} p_v \lambda^{k-v}\right] e_\lambda$$

and

$$\sum_{k=0}^{n} \alpha_k y^{\Delta^k} = \left(\sum_{v=0}^{n} p_v \frac{\varphi^{(v)}(\lambda)}{v!}\right) e_\lambda.$$

Proof.*: We use induction to prove the theorem. Take* $k = 0$. *Then*

$p_0 = p, \quad y(t) = p(t)e_\lambda(t, t_0)$.

We have

$$y^{\Delta^0} = y(t) = p_0 e_\lambda(t, t_0) = \left[\sum_{v=0}^{0} \begin{pmatrix} 0 \\ 0 \end{pmatrix} p_0 \lambda^0\right] e_\lambda.$$

Also, from $\varphi(\lambda) = \alpha_0$ we have

$$\alpha_0 y^{\Delta^0} = \left(\sum_{v=0}^{0} p_0 \frac{\varphi^{(0)}(\lambda)}{0!}\right) e_\lambda = \alpha_0 p_0 e_\lambda.$$

Assume that for some $k \in \mathbb{N}_0$ we have

$$y^{\Delta^k} = \left[\sum_{v=0}^{k} \binom{k}{v} p_v \lambda^{k-v}\right] e_\lambda,$$

and

$$\sum_{k=0}^{n} \alpha_k y^{\Delta^k} = \left(\sum_{v=0}^{n} p_v \frac{\varphi^{(v)}(\lambda)}{v!}\right) e_\lambda.$$

We will show the equalities for $k+1$.

$$\begin{aligned}
y^{\Delta^{k+1}} &= \left(y^{\Delta^k}\right)^\Delta = \left\{\left[\sum_{v=0}^{k} \binom{k}{v} p_v \lambda^{k-v}\right] e_\lambda\right\}^\Delta \\
&= \sum_{v=0}^{k} \binom{k}{v} \lambda^{k-v} (p_v e_\lambda)^\Delta = \sum_{v=0}^{k} \binom{k}{v} \lambda^{k-v} \left[p_v^\Delta e_\lambda^\sigma + p_v \lambda e_\lambda\right] \\
&= \sum_{v=0}^{k} \binom{k}{v} \lambda^{k-v} \left[p_v \lambda e_\lambda + p_v^\Delta (1+\mu\lambda) e_\lambda\right] = \sum_{v=0}^{k} \binom{k}{v} \lambda^{k-v} (\lambda p_v + p_{v+1}) e_\lambda \\
&= \sum_{v=0}^{k} \binom{k}{v} \lambda^{k-v+1} p_v e_\lambda + \sum_{v=0}^{k} \binom{k}{v} \lambda^{k-v} p_{v+1} e_\lambda \\
&= \sum_{v=0}^{k} \binom{k}{v} \lambda^{k-v+1} p_v e_\lambda + \sum_{v=1}^{k+1} \binom{k}{v-1} \lambda^{k-v+1} p_v e_\lambda \\
&= \binom{k}{0} \lambda^{k+1} p_0 e_\lambda + \sum_{v=1}^{k} \left[\binom{k}{v} + \binom{k}{v-1}\right] \lambda^{k-v+1} p_v e_\lambda + \binom{k}{k} \lambda^0 p_{k+1} e_\lambda \\
&= \left[\binom{k+1}{0} \lambda^{k+1} p_0 + \sum_{v=1}^{k} \binom{k+1}{v} \lambda^{k+1-v} p_v + \binom{k+1}{k+1} p_{k+1}\right] e_\lambda \\
&= \left[\sum_{v=0}^{k+1} \lambda^{k+1-v} \binom{k+1}{v} p_v\right] e_\lambda.
\end{aligned}$$

The second identity follows easily from the fact that

$$\varphi(\lambda) = \sum_{k=0}^{n} \alpha_k \lambda^k \Rightarrow \varphi^{(v)}(\lambda) = \sum_{k=v}^{n} \alpha_k k(k-1)\dots(k-v+1)\lambda^{k-v}.$$

Theorem 3.2.7 *Let λ be a zero of φ of multiplicity at least $m \in \mathbb{N}$ and let $\lambda \in \mathbb{R}$. Define*

$$p_{m-1} = 1, p_k = 0 \ \textit{for all } k \geq m$$

and

$$p_k(t) = \int_{t_0}^{t} \frac{p_{k+1}(\tau)}{1+\lambda\mu(\tau)}\Delta\tau \ \ for\ 0 \le k \le m-2.$$

Then $y(t) = p(t)e_\lambda(t,t_0)$ *is a solution where* $p = p_0$.

Example 3.2.8 *If* λ *is a double zero of* φ, *then*

$$y_1(t) = e_\lambda(t,t_0) \ \ and\ \ y_2(t) = e_\lambda(t,t_0)\int_{t_0}^{t}\frac{\Delta\tau}{1+\lambda\mu(\tau)}$$

are two solutions of the linear equation (3.9).

3.3 Linear higher order nonhomogeneous dynamic equations

In this section we give basic theoretical details related with higher order linear dynamic equations.

Definition 3.3.1 *[5] Let* $x_1, x_2, x_3, \ldots, x_n : \mathbb{T} \to \mathbb{R}$ *be given functions. The Wronskians detrminant of* $x_1, x_2, x_3, \ldots, x_n$ *is defined as:*

$$W(x_1,\ldots,x_n) = det\begin{pmatrix} x_1 & \ldots & x_n \\ x_1^{\Delta} & \ldots & x_n^{\Delta} \\ x_1^{\Delta^{n-1}} & \ldots & x_n^{\Delta^{n-1}} \end{pmatrix}.$$

Consider the Initial Value Problem for second order linear nonhomogeneous dynamic equation with

$$y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y = f(t), \ \ y(t_0) = y_0, \ \ y^{\Delta}(t_0) = y_0^{\Delta}. \tag{3.11}$$

Then we have the following theorem for its solution.

Theorem 3.3.2 *[5] If* f *is rd-continuous and if* y_1 *and* y_2 *form a fundamental system for the equation* $y^{\Delta\Delta} + p(t)y^{\Delta} + q(t)y = 0$, *then the solution of (3.11) is given by*

$$c_1y_1(t) + c_2y_2(t) + \int_{t_0}^{t}\frac{y_2(t)y_1(\sigma(\tau)) - y_1(t)y_2(\sigma(\tau))}{W(y_1,y_2)(\sigma(t_0))}f(\tau)\Delta\tau,$$

where

$$c_1 = \frac{y_2^{\Delta}(t_0)y_0 - y_2(t_0)y_0^{\Delta}}{W(y_1,y_2)(t_0)} \ \ and\ \ c_2 = \frac{y_1(t_0)y_0^{\Delta} - y_1^{\Delta}(t_0)y_0}{W(y_1,y_2)(t_0)}.$$

Here $W(y_1,y_2)(t_0)$ *is the Wronskian determinant of* y_1 *and* y_2.

For an n-th order nonhomogeneous dynamic equation

$$y^{\Delta^n} + \sum_{k=1}^{n} p_k(t) y^{\Delta^{n-k}} = f(t) \tag{3.12}$$

the general solution is given by the following theorem.

Theorem 3.3.3 *[5] If $f, p_1, \ldots, p_n$ are rd-continuous, $\sum_{k=1}^{n} (-\mu)^{k-1} p_k$ is regressive, and $y_1, \ldots, y_n$ is a fundamental system for $y^{\Delta^n} + \sum_{k=1}^{n} p_k(t) y^{\Delta^{n-k}} = 0$, then all solutions of (3.12) are given by*

$$\sum_{k=1}^{n} c_k y_k(t) + \int_{t_0}^{t} \frac{W(\sigma(\tau), t)}{W(\sigma(\tau))} f(\tau) \Delta\tau,$$

where $c_1, \ldots, c_n$ are constants, $W(\sigma(t)) = W(y_1, \ldots, y_n)$ and $W(\sigma(\tau, t))$ is the determinant of the matrix where the last row is replaced by $(y_1(t) \ldots y_n(t))$.

CHAPTER 4

SERIES SOLUTION METHOD

In theory of differential equations the series solution method is very important. As a matter of fact, it is one of the mostly used method to deal with linear nonconstant coefficient differential equations of higher order. Moreover, it can be applied to certain nonlinear equations as well.
Although the series solution method for differential equations is a very standard one, there are very few studies in the literature on series solutions of dynamic equations[36, 35, 27]. Moreover they are related with the case of constant graininess function.
In this chapter we discuss the details of a series solution method for dynamic equations on arbitrary time scales. We begin with some properties of the monomials $h_k(t, \alpha)$ introduced in Chapter 2.

4.1 Properties of the monomials $h_n(t, \alpha)$.

We first recall the Leibnitz formula for n-th order delta derivative of a product of two functions

Theorem 4.1.1 *[26] (Leibnitz Formula).*
Let $S_k^{(n)}$ be the set consisting of all possible strings of length n, containing exactly k times σ and $n-k$ times Δ. If f^{Λ} exists for all $\Lambda \in S_k^{(n)}$,

then

$$(fg)^{\Delta^n} = \sum_{k=0}^{n} \left(\sum_{\Lambda \in S_k^{(n)}} f^{\Lambda} \right) g^{\Delta^k}.$$

Theorem 4.1.2 *[26] For every $m, n \in \mathbb{N}_0$ we have*

$$h_n(t,\alpha)h_m(t,\alpha) = \sum_{l=m}^{m+n}\left(\sum_{\Lambda_{l,m}\in S_m^{(l)}} h_n^{\Lambda_{l,m}}(\alpha,\alpha)\right)h_l(t,\alpha)$$

for every $t, \alpha \in \mathbb{T}$.

Proof.

Consider first the case $m = 0$ or $n = 0$. Without loss of generality, assume that $n = 0$.

$$\begin{aligned} h_0(t,\alpha)h_m(t,\alpha) &= h_m(t,\alpha) = \sum_{l=m}^{m}\left(\sum_{\Lambda_{l,m}\in S_m^{(l)}} h_0^{\Lambda_{l,m}}(\alpha,\alpha)\right)h_l(t,\alpha) \\ &= \left(\sum_{\Lambda_{m,m}\in S_m^{(m)}} h_0^{\Lambda_{m,m}}(\alpha,\alpha)\right)h_m(t,\alpha) = h_m(t,\alpha). \end{aligned}$$

Now assume that $m \neq 0$ and $n \neq 0$. By Taylor's formula we have

$$h_n(t,\alpha)h_m(t,\alpha) = \sum_{l=0}^{\infty}(h_n(t,\alpha)h_m(t,\alpha))^{\Delta^l}\Big|_{t=\alpha} h_l(t,\alpha).$$

Applying Leibnitz's formula we obtain

$$(h_n(t,\alpha)h_m(t,\alpha))^{\Delta^l} = \sum_{k=0}^{l}\left(\sum_{\Lambda_{l,k}\in S_k^{(l)}} h_n^{\Lambda_{l,k}}(t,\alpha)\right)h_m^{\Delta^k}(t,\alpha).$$

If $l < m$, then

$$(h_n(t,\alpha)h_m(t,\alpha))^{\Delta^l} = \sum_{k=0}^{l}\left(\sum_{\Lambda_{l,k}\in S_k^{(l)}} h_n^{\Lambda_{l,k}}(t,\alpha)\right)h_{m-k}(t,\alpha).$$

We have $h_{m-k}(\alpha,\alpha) = 0$ and

$$(h_n(t,\alpha)h_m(t,\alpha))^{\Delta^l}\Big|_{t=\alpha} = 0.$$

If $l \geq m, h_0(t,\alpha) = 1$, and we get

$$\begin{aligned} (h_n(t,\alpha)h_m(t,\alpha))^{\Delta^l}\Big|_{t=\alpha} &= \textstyle\sum_{k=0}^{m-1}\left(\sum_{\Lambda_{l,k}\in S_k^{(l)}} h_n^{\Lambda_{l,k}}(t,\alpha)\right)h_{m-k}(t,\alpha)\Big|_{t=\alpha} \\ &+ \textstyle\sum_{\Lambda_{l,m}\in S_m^{(l)}} h_n^{\Lambda_{l,m}}(t,\alpha)\Big|_{t=\alpha} \\ &= \textstyle\sum_{\Lambda_{l,m}\in S_m^{(l)}} h_n^{\Lambda_{l,m}}(\alpha,\alpha). \end{aligned}$$

Hence, we conclude

$$\left.(h_n(t,\alpha)h_m(t,\alpha))^{\Delta^l}\right|_{t=\alpha}=\begin{cases}0 & \text{if } \ l<m\\ \sum_{\Lambda_{l,m}\in S_m^{(l)}} h_n^{\Lambda_{l,m}}(\alpha,\alpha) & \text{if } \ l\geq m\end{cases},$$

which results in

$$\begin{aligned}h_n(t,\alpha)h_m(t,\alpha) &= \sum_{l=m}^{\infty}\left.(h_n(t,\alpha)h_m(t,\alpha))^{\Delta^l}\right|_{t=\alpha}h_l(t,\alpha)\\ &= \sum_{l=m}^{\infty}\left(\sum_{\Lambda_{l,m}\in S_m^{(l)}} h_n^{\Lambda_{l,m}}(\alpha,\alpha)\right)h_l(t,\alpha)\\ &= \sum_{l=m}^{m+n}\left(\sum_{\Lambda_{l,m}\in S_m^{(l)}} h_n^{\Lambda_{l,m}}(\alpha,\alpha)\right)h_l(t,\alpha),\end{aligned}$$

so, the proof is completed.

Remark 4.1.3 *Using the notation*

$$D_{n,m,l}(\alpha)=\sum_{\Lambda_{l,m}\in S_m^{(l)}} h_n^{\Lambda_{l,m}}(\alpha,\alpha), \tag{4.1}$$

for $m,n\in\mathbb{N}_0$ and $l=m,m+1,\dots,m+n$, we have

$$h_n(t,\alpha)h_m(t,\alpha)=\sum_{l=m}^{m+n}D_{n,m,l}(\alpha)h_l(t,\alpha).$$

Theorem 4.1.4 *The Cauchy product of series on time scales is in the form*

$$\left(\sum_{i=0}^{\infty}A_ih_i(t,\alpha)\right)\left(\sum_{j=0}^{\infty}B_jh_j(t,\alpha)\right)=\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}A_lB_{k-1}\left(\sum_{r=k-l}^{k}D_{l,k-l,r}(\alpha)h_r(t,\alpha)\right)\right),$$

where A_i, B_j, are constants for $i\in\mathbb{N}_0$.

Proof.

By the Cauchy product of two infinite series, we get

$$\left(\sum_{i=0}^{\infty}A_ih_i(t,\alpha)\right)\left(\sum_{j=0}^{\infty}B_jh_j(t,\alpha)\right)=\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}A_lh_l(t,\alpha)B_{k-l}h_{k-l}(t,\alpha)\right).$$

By the Theorem 4.1.2 we have

$$\begin{aligned}\left(\sum_{i=0}^{\infty}A_ih_i(t,\alpha)\right)\left(\sum_{j=0}^{\infty}B_jh_j(t,\alpha)\right) &= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}A_lB_{k-l}\left(\sum_{r=k-l}^{k}\left(\sum_{\Lambda_{r,k-l}\in S_{k-l}^{(r)}} h_l^{\Lambda_{r,k-l}}(\alpha,\alpha)\right)h_r(t,\alpha)\right)\right)\\ &= \sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}A_lB_{k-1}\left(\sum_{r=k-l}^{k}D_{l,k-l,r}(\alpha)h_r(t,\alpha)\right)\right),\end{aligned}$$

which completes the proof.

4.2 Series solution method for dynamic equations

Let $\mathbb{T}$ be a time scale with forward jump operator σ and delta derivative operator Δ . Let $\alpha \in \mathbb{T}$.

Consider the linear dynamic equation

$$y^{\Delta^n}(t) + a_1(t)y^{\Delta^{n-1}}(t) + ... + a_{n-1}(t)y^{\Delta}(t) + a_n(t)y(t) = f(t), \tag{4.2}$$

where

$$a_m(t) = \sum_{i=0}^{\infty} a_i^m h_i(t,\alpha), \qquad f(t) = \sum_{i=0}^{\infty} f_i h_i(t,\alpha). \tag{4.3}$$

Assume that the solution $y(t)$ is in the form

$$y(t) = \sum_{i=0}^{\infty} c_i h_i(t,\alpha), \tag{4.4}$$

where c_i are cofficients to be determined from the dynamic equation.

We have

$$\begin{aligned} y^{\Delta^r}(t) &= \sum_{i=0}^{\infty} c_i h_i^{\Delta^r}(t,\alpha) \\ &= \sum_{i=r}^{\infty} c_i h_{i-r}(t,\alpha) \\ &= \sum_{i=0}^{\infty} c_{i+r} h_i(t,\alpha), \end{aligned}$$

for $r = 1, \dots, n$. By Theorem 4.1.4 we obtain

$$\begin{aligned} a_m(t)y^{n-m}(t) &= \left(\sum_{i=0}^{\infty} a_i^m h_i(t,\alpha)\right)\left(\sum_{j=0}^{\infty} c_{j+n-m} h_j(t,\alpha)\right) \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{k} a_l^m c_{k-l+n-m}\left(\sum_{r=k-l}^{k} D_{k,k-l,r}(\alpha)\right) h_r(t,\alpha) \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{r=k-l}^{k} a_l^m c_{k-l+n-m} D_{k,k-l,r}(\alpha) h_r(t,\alpha), \end{aligned} \tag{4.5}$$

where

$$D_{k,k-l,r}(\alpha) = \sum_{\Lambda_{r,k-l} \in S_{k-l}^{(r)}} h_r^{\Lambda_{k,k-l}}(\alpha,\alpha).$$

We rewrite the inner double sum in (4.5) as

$$\sum_{l=0}^{k}\sum_{r=k-l}^{k} = \sum_{r=0}^{k}\sum_{l=k-r}^{k}.$$

Hence, we obtain

$$\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{r=k-l}^{k} = \sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{l=k-r}^{k}.$$

Now rewrite the outer double sum in the above formula as

$$\sum_{k=0}^{\infty}\sum_{r=0}^{k} = \sum_{r=0}^{\infty}\sum_{k=r}^{\infty}.$$

Then we conclude that

$$\sum_{k=0}^{\infty}\sum_{r=0}^{k}\sum_{l=k-r}^{k} = \sum_{r=0}^{\infty}\sum_{k=r}^{\infty}\sum_{l=k-r}^{k}.$$

Now the last row in (4.5) becomes

$$a_m(t)y^{n-m}(t) = \sum_{r=0}^{\infty}\left(\sum_{k=r}^{\infty}\sum_{l=k-r}^{k} a_l^m c_{k-l+n-m} D_{k,k-l,r}(\alpha)\right) h_r(t,\alpha). \tag{4.6}$$

Put (4.6) into the equation (4.2)

$$\begin{aligned}\sum_{r=0}^{\infty} c_{r+n}h_r(t,\alpha) \quad &+ \quad \sum_{r=0}^{\infty}\left(\sum_{k=r}^{\infty}\sum_{l=k-r}^{k} a_l^1 c_{k-l+n-1} D_{k,k-l,r}(\alpha)\right) h_r(t,\alpha)\\ &+ \quad \sum_{r=0}^{\infty}\left(\sum_{k=r}^{\infty}\sum_{l=k-r}^{k} a_l^2 c_{k-l+n-2} D_{k,k-l,r}(\alpha)\right) h_r(t,\alpha)\\ &+ \quad ...... + \sum_{r=0}^{\infty}\left(\sum_{k=r}^{\infty}\sum_{l=k-r}^{k} a_l^n c_{k-l} D_{k,k-l,r}(\alpha)\right) h_r(t,\alpha)\\ &= \quad \sum_{r=0}^{\infty} f_r h_r(t,\alpha).\end{aligned}$$

This implies

$$\begin{aligned}&\sum_{r=0}^{\infty}\left\{c_{r+n} + \sum_{k=r}^{\infty}\sum_{l=k-r}^{k} D_{k,k-l,r}(\alpha)\left(a_l^1 c_{k-l+n-1} + a_l^2 c_{k-l+n-2} + \cdots + a_l^n c_{k-l}\right)\right\} h_r(t,\alpha)\\ &= \sum_{r=0}^{\infty} f_r h_r(t,\alpha).\end{aligned}$$

We equate both sides to get

$$c_{r+n} = f_r - \sum_{k=r}^{\infty}\sum_{l=k-r}^{k} D_{k,k-l,r}(\alpha)\left(a_l^1 c_{k-l+n-1} + a_l^2 c_{k-l+n-2} + \cdots + a_l^n c_{k-l}\right), \tag{4.7}$$

for $r = 0, 1, 2, \ldots$, which is the general recurrence relation for an n-th order linear dynamic equation with nonconstant coefficients on arbitrary time scales $\mathbb{T}$.

4.3 Properties of the constants $D_{n,m,l}(\alpha)$.

The constants $D_{n,m,l}(\alpha)$ which have been defined in (4.1) have some properties which we discuss below.

(1) For all $m \in \mathbb{N}_0$ and $l = 0, \ldots, m$, we have

$$D_{0,m,l}(\alpha) = D_{m,0,l}(\alpha) = \begin{cases} 0, & \text{if } \ l \neq m \\ 1, & \text{if } \ l = m \end{cases} . \tag{4.8}$$

Proof.

This statement easily follows from the fact that

$$h_0(t,\alpha)h_m(t,\alpha) = h_m(t,\alpha)h_0(t,\alpha) = h_m(t,\alpha).$$

Indeed,

$$h_m(t,\alpha) = h_0(t,\alpha)h_m(t,\alpha) = \sum_{l=m}^{m} D_{0,m,l}(\alpha)h_l(t,\alpha) = D_{0,m,m}(\alpha)h_m(t,\alpha),$$

implies $D_{0,m,m}(\alpha) = 1$ and

$$h_m(t,\alpha) = h_m(t,\alpha)h_0(t,\alpha) = \sum_{l=0}^{m} D_{m,0,l}(\alpha)h_l(t,\alpha),$$

implies $D_{m,0,m}(\alpha) = 1$ and $D_{m,0,l}(\alpha) = 0$ for $l = 0, ..., m-1$.

(2) For all $n, m \in \mathbb{N}_0$ and $l = m, ..., n+m$, we have

$$D_{n,m,l}(\alpha) = D_{m,n,l}(\alpha) \text{ for } n \geq m \text{ and } l \geq n$$

$$D_{n,m,l}(\alpha) = D_{m,n,l}(\alpha) = 0 \text{ for } n \geq m \text{ and } m \leq l < n \tag{4.9}$$

Proof.

Since

$$h_n(t,\alpha)h_m(t,\alpha) = h_m(t,\alpha)h_n(t,\alpha),$$

we have

$$\sum_{l=m}^{m+n} D_{n,m,l}(\alpha)h_l(t,\alpha) = \sum_{l=n}^{m+n} D_{m,n,l}(\alpha)h_l(t,\alpha)$$

from which it follows that if $n \geq m$ we should have

$D_{n,m,m}(\alpha) = D_{n,m,m+1}(\alpha) = ... = D_{n,m,n-1}(\alpha) = 0$

and also $D_{n,m,l}(\alpha) = D_{m,n,l}(\alpha)$ for $l = n, n+1, ..., n+m$.

We next discuss the computation of the constants $D_{n,m,l}(\alpha)$ on arbitrary time scales. Let $\mathbb{T}$ be a time scale with the forward jump operator $\sigma(t)$, delta differentiation operator Δ and the graininess function $\mu(t)$. Let $h_l(t,\alpha), l = 0,1,2,3$ be the first four monomials on $\mathbb{T}$. We are using the following relations in the computations of the constants $D_{n,m,l}(\alpha)$.

$h_k^{\Delta}(t,\alpha) = h_{k-1}(t,\alpha)$

$$\begin{aligned} h_k^{\sigma}(t,\alpha) &= h_k(t,\alpha) + \mu(t)h_k^{\Delta}(t,\alpha) \\ &= h_k(t,\alpha) + \mu(t)h_{k-1}(t,\alpha) \end{aligned}$$

$$\begin{aligned} h_k^{\Delta\sigma}(t,\alpha) &= h_{k-1}^{\sigma}(t,\alpha) \\ &= h_{k-1}(t,\alpha) + \mu(t)h_{k-2}(t,\alpha) \end{aligned}$$

$h_k^{\sigma\Delta}(t,\alpha) = (1+\mu^{\Delta}(t))h_k^{\Delta\sigma}(t,\alpha)$

$$\begin{aligned} h_k^{\sigma\sigma}(t,\alpha) &= h_k(t,\alpha) + \mu(t)h_k^{\Delta}(t,\alpha) + \mu^{\sigma}(t)[h_k^{\Delta}(t,\alpha) + \mu(t)k_k^{\Delta^2}(t,\alpha)] \\ &= h_k(t,\alpha) + (\mu(t)+\mu^{\sigma}(t))h_{k-1}(t,\alpha) + \mu^{\sigma}(t)\mu(t)h_{k-2}(t,\alpha), \end{aligned}$$

$h_k^{\Delta\Delta}(t,\alpha) = h_{k-2}(t,\alpha)$

$$\begin{aligned} h_k^{\sigma\sigma\sigma}(t,\alpha) &= (h_k(t,\alpha) + \mu(t)h_{k-1}(t,\alpha))^{\sigma\sigma} \\ &= h_k(t,\alpha) \\ &+ [\mu(t) + (\mu(t)+\mu^{\sigma}(t))(1+\mu^{\Delta}(t)) + \mu(t)\mu^{\sigma}(t)\mu^{\Delta\Delta}(t)]h_{k-1}(t,\alpha) \\ &+ [\mu^{\sigma}(t)(\mu(t)+\mu^{\sigma}(t)) + \mu(t)\mu^{\sigma}(t)(1+\mu^{\Delta\sigma}(t)+\mu^{\sigma\Delta}(t))]h_{k-2}(t,\alpha) \\ &+ \mu^{\sigma\sigma}(t)\mu^{\sigma}(t)\mu(t)h_{k-3}(t,\alpha). \end{aligned}$$

Then we compute the following.

(1) Let $l = 0$. Then from the property (4.8) we have

$$D_{0,0,0}(\alpha) = 1.$$

(2) Let $l = 1$. Then from the property (4.8) we have

$$D_{1,0,1}(\alpha) = D_{0,1,1}(\alpha) = 1,$$

and

$$D_{1,1,1}(\alpha) = h_1^{\sigma}(\alpha,\alpha) + \mu(\alpha)h_0(\alpha,\alpha) = \mu(\alpha).$$

(3) Let $l = 2$. Then we have

$$D_{2,0,2}(\alpha) = D_{0,2,2}(\alpha) = 1.$$

We also compute

$$\begin{aligned}
D_{1,1,2}(\alpha) &= h_1^{\sigma\Delta}(\alpha,\alpha) + h_1^{\Delta\sigma}(\alpha,\alpha)\\
&= (2+\mu^{\Delta}(\alpha))h_1^{\Delta\sigma}(\alpha,\alpha)\\
&= (2+\mu^{\Delta}(\alpha))h_0^{\sigma}(\alpha,\alpha)\\
&= 2+\mu^{\Delta}(\alpha),
\end{aligned}$$

$$\begin{aligned}
D_{2,1,2}(\alpha) = D_{1,2,2}(\alpha) &= h_1^{\sigma\sigma}(\alpha,\alpha)\\
&= h_1(\alpha,\alpha) + (\mu(\alpha)+\mu^{\sigma}(\alpha))h_0(\alpha,\alpha)\\
&= \mu(\alpha)+\mu^{\sigma}(\alpha),
\end{aligned}$$

and

$$\begin{aligned}
D_{2,2,2}(\alpha) &= h_2^{\sigma\sigma}(\alpha,\alpha)\\
&= h_2(\alpha,\alpha) + (\mu(\alpha)+\mu^{\sigma}(\alpha))h_1(\alpha,\alpha) + \mu(\alpha)\mu^{\sigma}(\alpha)h_0(\alpha,\alpha)\\
&= \mu(\alpha)\mu^{\sigma}(\alpha).
\end{aligned}$$

(4) Let $l = 3$. Then we compute

$$D_{3,0,3}(\alpha) = D_{0,3,3}(\alpha) = 1,$$

$$\begin{aligned}
D_{1,1,3}(\alpha) &= h_1^{\Delta\Delta\sigma}(\alpha,\alpha) + h_1^{\Delta\sigma\Delta}(\alpha,\alpha) + h_1^{\sigma\Delta\Delta}(\alpha,\alpha)\\
&= \mu^{\Delta\Delta}(\alpha),
\end{aligned}$$

$$\begin{aligned}
D_{1,2,3}(\alpha) &= D_{2,1,3}(\alpha) = h_2^{\Delta\Delta\sigma}(\alpha,\alpha) + h_2^{\Delta\sigma\Delta}(\alpha,\alpha) + h_2^{\sigma\Delta\Delta}(\alpha,\alpha)\\
&= h_0^{\sigma}(\alpha,\alpha) + h_1^{\sigma\Delta}(\alpha,\alpha) + (h_2(\alpha,\alpha)+\mu(\alpha)h_1(\alpha,\alpha))^{\Delta\Delta}\\
&= (2+\mu^{\Delta}(\alpha))h_0^{\sigma}(\alpha,\alpha)\\
&= 3+\mu^{\Delta}(\alpha)+\mu^{\Delta\sigma}(\alpha)+\mu^{\Delta\sigma}(\alpha)+\mu^{\sigma\Delta)}(\alpha),
\end{aligned}$$

$$\begin{aligned}
D_{2,2,3}(\alpha) &= h_2^{\Delta\sigma\sigma}(\alpha,\alpha) + h_2^{\sigma\Delta\sigma}(\alpha,\alpha) + h_2^{\sigma\sigma\Delta}(\alpha,\alpha)\\
&= 2(\mu(\alpha)+\mu^{\sigma}) + (\mu(\alpha)+\mu^{\sigma}(\alpha))^{\sigma}\\
&+ (\mu(\alpha)\mu^{\sigma}(\alpha))^{\Delta} + \mu^{\Delta\sigma}(\alpha)(2+\mu^{\Delta}(\alpha)),
\end{aligned}$$

$$\begin{aligned}
D_{3,1,3}(\alpha) &= D_{1,3,3}(\alpha)\\
&= h_1^{\sigma\sigma\sigma}(\alpha,\alpha)\\
&= h_1(\alpha,\alpha)\\
&+ [\mu(\alpha) + (\mu(\alpha)+\mu^{\sigma}(\alpha))(1+\mu^{\Delta}(\alpha)) + \mu(\alpha)\mu^{\sigma}(\alpha)\mu^{\Delta\Delta}(\alpha)]h_0(\alpha,\alpha)\\
&= \mu(\alpha) + (\mu(\alpha)+\mu^{\sigma}(\alpha))(1+\mu^{\Delta}(\alpha)) + \mu(\alpha)\mu^{\sigma}(\alpha)\mu^{\Delta\Delta}(\alpha),
\end{aligned}$$

$$
\begin{aligned}
D_{3,2,3}(\alpha) &= D_{2,3,3}(\alpha) \\
&= h_2^{\sigma\sigma\sigma}(\alpha,\alpha) \\
&= h_2(\alpha,\alpha) \\
&+ [\mu(\alpha) + (\mu(\alpha) + \mu^{\sigma}(\alpha))(1 + \mu^{\Delta}(\alpha)) + \mu(\alpha)\mu^{\sigma}(\alpha)\mu^{\Delta\Delta}(\alpha)]h_1(\alpha,\alpha) \\
&+ [\mu^{\sigma}(\alpha)(\mu(\alpha) + \mu^{\sigma}(\alpha)) + \mu(\alpha)\mu^{\sigma}(\alpha)(1 + \mu^{\Delta\sigma}(\alpha) + \mu^{\sigma\Delta}(\alpha))]h_0(\alpha,\alpha) \\
&= \mu^{\sigma}(\alpha)(\mu(\alpha) + \mu^{\sigma}(\alpha)) + \mu(\alpha)\mu^{\sigma}(\alpha)(1 + \mu^{\Delta\sigma}(\alpha) + \mu^{\sigma\Delta}(\alpha)),
\end{aligned}
$$

$$
\begin{aligned}
D_{3,3,3}(\alpha) &= h_3^{\sigma\sigma\sigma}(\alpha,\alpha) \\
&= h_3(\alpha,\alpha) \\
&+ [\mu(\alpha) + (\mu(\alpha) + \mu^{\sigma}(\alpha))(1 + \mu^{\Delta}(\alpha)) + \mu(\alpha)\mu^{\sigma}(\alpha)\mu^{\Delta\Delta}(\alpha)]h_2(\alpha,\alpha) \\
&+ [\mu^{\sigma}(\alpha)(\mu(\alpha) + \mu^{\sigma}(\alpha)) + \mu(\alpha)\mu^{\sigma}(\alpha)(1 + \mu^{\Delta\sigma}(\alpha) + \mu^{\sigma\Delta}(\alpha))]h_1(\alpha,\alpha) \\
&+ \mu(\alpha)\mu^{\sigma}(\alpha)\mu\sigma\sigma(\alpha)h_0(\alpha,\alpha) \\
&= \mu(\alpha) + \mu^{\sigma}(\alpha)\mu^{\sigma\sigma}(\alpha).
\end{aligned}
$$

CHAPTER 5

APPLICATION OF THE SERIES SOLUTION METHOD

In this chapter we apply the series solution method given in Chapter 4 to some particular examples of dynamic equations. We first test the method on constant coefficient equation whose solution is known.

5.1 Constant coefficient equations

Consider the second order constant coefficient dynamic equation

$$y^{\Delta\Delta} - 3y^{\Delta} + 2y = 0, \quad t \in \mathbb{T} = 2^{\mathbb{N}_0}.$$

Its characteristic equation is

$$\lambda^2 - 3\lambda + 2 = 0,$$

and its solutions are $\lambda = 1, \lambda = 2$.
In fact, this is the Example 3.2.5 and its solutions are obtained as $y_1(t) = e_1(t, 1), y_2(t) = e_2(t, 1)$.
We will apply the series solution method to this equation, that is,

$$y^{\Delta\Delta} - 3y^{\Delta} + 2y = 0, \quad t \in \mathbb{T} = 2^{\mathbb{N}_0}.$$

We take $\alpha = 1$ and propose a solution in the form

$$y(t) = \sum_{n=0}^{\infty} c_n h_n(t, 1).$$

Then

$$y^{\Delta}(t) = \sum_{n=0}^{\infty} c_{n+1} h_n(t, 1),$$

$$y^{\Delta\Delta}(t) = \sum_{n=0}^{\infty} c_{n+2} h_n(t,1).$$

Put $y(t), y^{\Delta}(t)$ and $y^{\Delta\Delta}(t)$ in to the equation

$$\sum_{n=0}^{\infty} c_{n+2} h_n(t,1) - 3\sum_{n=0}^{\infty} c_{n+1} h_n(t,1) + 2\sum_{n=0}^{\infty} c_n h_n(t,1) = 0,$$

or upon combining the series,

$$\sum_{n=0}^{\infty} (c_{n+2} - 3c_{n+1} + 2c_n) h_n(t,1) = 0.$$

Then we have the following recurrence relation,

$$c_{n+2} = 3c_{n+1} - 2c_n, \text{ for } n = 0, 1, 2, \ldots$$

We compute few terms as follows

$n = 0, \quad c_2 = 3c_1 - 2c_0,$

$n = 1, \quad c_3 = 3c_2 - 2c_1 = 7c_1 - 6c_0,$

$n = 2, \quad c_4 = 3c_3 - 2c_2 = 15c_1 - 14c_0,$

$n = 3, \quad c_5 = 3c_4 - 2c_3 = 31c_1 - 30c_0,$

$n = 4, \quad c_6 = 3c_5 - 2c_4 = 63c_1 - 62c_0,$

which can be generalized as follows

$$c_n = (2^n - 1)c_1 - (2^n - 2)c_0, \text{ for } n = 2, 3, 4, \ldots..$$

Then the solution $y(t)$ becomes

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} c_n h_n(t,1) \\ &= c_0 h_0 + c_1 h_1 + c_2 h_2 + c_3 h_3 + \ldots \\ &= c_0 h_0 + c_1 h_1 + \sum_{n=2}^{\infty} c_n h_n(t,1) \\ &= c_0 h_0 + c_1 h_1 + \sum_{n=2}^{\infty} [(2^n - 1)c_1 - (2^n - 2)c_0] h_n(t,1). \end{aligned}$$

We arrange the series of $y(t)$ as

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} c_n h_n(t,1) \\ &= c_0 h_0(t,1) + c_1 h_1(t,1) + (c_1 - c_0)\sum_{n=2}^{\infty} 2^n h_n(t,1) + (2c_0 - c_1)\sum_{n=2}^{\infty} h_n(t,1) \\ &= (c_1 - c_0)\sum_{n=0}^{\infty} 2^n h_n(t,1) + (2c_0 - c_1)\sum_{n=0}^{\infty} h_n(t,1) \end{aligned}$$

Since $e_\lambda(t,1) = \sum_{n=0}^{\infty} \lambda^n h_n(t,1)$, then we conclude

$$\begin{aligned} y(t) &= (c_1 - c_0)e_2(t,1) + (2c_0 - c_1)e_1(t,1) \\ &= d_1 e_2(t,1) + d_2 e_1(t,1), \end{aligned}$$

which is the general solution of the dynamic equation.

5.2 Non-constant coefficient equations

In this section we apply the series solution method to some dynamic equations with variable coefficients.

5.2.1 Example 1

As a first example we consider the equation

$$y^{\Delta\Delta} - \frac{t^2}{2} y^{\Delta} + \lambda \left(1 + \mu(t)\frac{t^2}{2}\right) y = 0, t \in \mathbb{T}. \tag{5.1}$$

We will apply the series solution method to (5.1). Let $\alpha \in \mathbb{T}$ and propose a series for $y(t)$

$$y(t) = \sum_{n=0}^{\infty} c_n h_n(t,\alpha) = c_0 h_0(t,\alpha) + c_1 h_1(t,\alpha) + \dots,$$

so that

$$y^{\Delta}(t) = \sum_{n=0}^{\infty} c_n h_n^{\Delta}(t,\alpha) = \sum_{n=1}^{\infty} c_n h_{n-1}(t,\alpha) = \sum_{n=0}^{\infty} c_{n+1} h_n(t,\alpha),$$

and

$$y^{\Delta^2}(t) = \sum_{n=1}^{\infty} c_{n+1} h_{n-1}(t,\alpha) = \sum_{n=0}^{\infty} c_{n+2} h_n(t,\alpha).$$

Suppose that

$$-\frac{t^2}{2} = \sum_{m=0}^{\infty} a_m h_m(t,\alpha)$$

and

$$1 + \mu(t)\frac{t^2}{2} = \sum_{m=0}^{\infty} b_m h_m(t,\alpha).$$

Then we put the series into the equation to obtain

$$\sum_{n=0}^{\infty} c_{n+2} h_n(t,\alpha) + \sum_{m=0}^{\infty} a_m h_m(t,\alpha) \sum_{n=0}^{\infty} c_{n+1} h_n(t,\alpha) + \lambda \sum_{m=0}^{\infty} b_m h_m(t,\alpha) \sum_{n=0}^{\infty} c_n h_n(t,\alpha) = 0$$

Using the Cauchy product for the series we obtain

$$\sum_{n=0}^{\infty} c_{n+2}h_n(t,\alpha)+\sum_{n=0}^{\infty}\sum_{m=0}^{n} a_m c_{n-m+1}h_m(t,\alpha)h_{n-m}(t,\alpha)+\lambda\sum_{n=0}^{\infty}\sum_{m=0}^{n} b_m c_{n-m}h_m(t,\alpha)h_{n-m}(t,\alpha)=0. \tag{5.2}$$

By the Theorem 4.1.2 we have

$$\begin{aligned} h_m(t,a)h_{n-m}(t,\alpha) &= \sum_{l=n-m}^{n}\left(\sum_{\Lambda_{l,n-m}\in S_{n-m}^{(l)}} h_m^{\Lambda_{l,n-m}}(\alpha,\alpha)\right)h_l(t,\alpha) \\ &= \sum_{l=n-m}^{n} D_{m,n-m,l}(\alpha)h_l(t,\alpha). \end{aligned}$$

Then we rewrite (5.2) as follows

$$\sum_{n=0}^{\infty} c_{n+2}h_n(t,a)+\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=n-m}^{n}(a_m c_{n-m+1}+\lambda b_m c_{n-m})D_{m,n-m,l}(\alpha)h_{m,l}(t,\alpha)=0. \tag{5.3}$$

As we did in the previous chapter, we change the order in the triple sum in the equation as

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=n-m}^{n}=\sum_{l=0}^{\infty}\sum_{n=l}^{\infty}\sum_{m=n-l}^{n},$$

so the equation (5.3) becomes

$$\sum_{l=0}^{\infty}\left[c_{l+2}+\sum_{n=l}^{\infty}\sum_{m=n-l}^{n}(a_m c_{n-m+1}+\lambda b_m c_{n-m})D_{m,n-m,l}(\alpha)\right]h_{m,l}(t,\alpha)=0. \tag{5.4}$$

As a result, the equation (5.4) yields the following recurrence relation

$$c_{l+2}=-\sum_{n=l}^{\infty}\sum_{m=n-l}^{n}(a_m c_{n-m+1}+\lambda b_m c_{n-m})D_{m,n-m,l}(\alpha),\ \ for\ l=1,2,\ldots \tag{5.5}$$

Example 5.2.1 *As a particular case we consider the dynamic equation (5.1) on the time scale $\mathbb{T}=a\mathbb{N}_0$. On this time scale the forward jump operator $\sigma(t)$ and the graininess function $\mu(t)$ are defined as $\sigma(t)=t+a$, and $\mu(t)=a$ respectively. Then the dynamic equation becomes*

$$y^{\Delta\Delta}-\frac{t^2}{2}y^{\Delta}+\lambda\left(1+\frac{at^2}{2}\right)y=0. \tag{5.6}$$

Taking $\alpha=0$ we compute the first three monomials $h_k(t,0)$, $\quad k=0,1,2$ on $a\mathbb{N}_0$ as

$$h_0(t,0)=1,$$

$$h_1(t,0)=\int_0^t h_0(\tau,0)\Delta\tau=t-0=t,$$

$$h_2(t,0) = \int_0^t h_1(\tau,0)\Delta\tau = \int_0^t \tau\Delta\tau = \frac{t(t-a)}{2}.$$

Then we have

$$-\frac{t^2}{2} = a_0h_0(t,0) + a_1h_1(t,0) + a_2h_2(t,0) = a_0 + a_1t + a_2\left(\frac{t^2}{2} - \frac{at}{2}\right),$$

and

$$1 + \frac{at^2}{2} = b_0h_0(t,0) + b_1h_1(t,0) + b_2h_2(t,0) = b_0 + b_1t + b_2\left(\frac{t^2}{2} - \frac{at}{2}\right),$$

that is, $a_0 = 0, a_1 = -\frac{a}{2}, a_2 = -1, b_0 = 1, b_1 = \frac{a^2}{2}$ *and* $b_2 = a$. *The recurrence relation for this particular time scale becomes,*

$$c_{l+2} = -\sum_{n=l}^{\infty}\sum_{m=n-l}^{n}(a_mc_{n-m+1} + \lambda b_mc_{n-m})D_{m,n-m,l}(0), l = 0,1,2,......$$

Therefore, we obtain

$$\begin{aligned} c_2 &= -(a_0c_1 + \lambda b_0c_0)D_{0,0,0}(0),\\ c_3 &= -(a_0c_2 + \lambda b_0c_1)D_{0,1,1}(0) - (a_1c_1 + \lambda b_1c_0)D_{1,0,1}(0) - (a_1c_2 + \lambda b_1c_1)D_{1,1,1}(0), \end{aligned}$$

and for $l = 2,3,4,.....$ *we have*

$$\begin{aligned} c_{l+2} &= -(a_0c_{l+1} + \lambda b_0c_l)D_{0,l,l}(0) - (a_1c_l + \lambda b_1c_{l-1})D_{1,l-1,l}(0) - (a_2c_{l-1} + \lambda b_2c_{l-2})D_{2,l-2,l}(0)\\ &- (a_1c_{l+1} + \lambda b_1c_l)D_{1,l,l}(0) - (a_2c_l + \lambda b_2c_{l-1})D_{2,l-1,l}(0) - (a_2c_{l+1} + \lambda b_2c_l)D_{2,l,l}(0). \end{aligned}$$

For this particular time scale $\mu(t) = a, \mu^\Delta(t) = 0, \mu^\sigma(t) = \mu(t+a) = a$. *Therefore*

$D_{0,0,0}(0) = D_{1,0,1}(0) = D_{0,1,1}(0) = D_{2,0,2}(0) = D_{0,2,2}(0) = 1,$

$D_{1,1,1}(0) = \mu(0) = a,$

$D_{1,1,2}(0) = 2 + \mu^\Delta(0) = 2,$

$D_{2,1,2}(0) = D_{1,2,2}(0) = \mu(0) + \mu^\sigma(0) = 2a,$

$D_{2,2,2}(0) = \mu(0)\mu^\sigma(0) = a^2.$

Hence, we obtain

$$\begin{aligned} c_2 &= -\lambda c_0\\ c_3 &= \frac{a^2}{2}c_2 + \left[\frac{a}{2} - \lambda\left(1 + \frac{a^3}{2}\right)\right]c_1 - \frac{\lambda a^2}{2}c_0 \end{aligned}$$

$$c_4 = 2a^2c_3 + (3a - \lambda(1 + 2a^3))c_2 + (1 - 3\lambda a^2)c_1 - \lambda a c_0.$$

For $l = 3, 4, ...$ we conclude

$$\begin{aligned} c_{l+2} &= \left[\frac{a}{2}D_{1,l,l}(0) + D_{2,l,l}(0)\right]c_{l+1} \\ &+ \left[\frac{a}{2}D_{1,l-1,l}(0) + D_{2,l-1,l}(0) - \lambda\left(D_{0,l,l} + \frac{a^2}{2}D_{1,l,l}(0) + aD_{2,l,l}(0)\right)\right]c_l \\ &+ \left[D_{2,l-2,l}(0) - \lambda\left(\frac{a^2}{2}D_{1,l-1,l}(0) + aD_{2,l-1,l}(0)\right)\right]c_{l-1} \\ &- \lambda a D_{2,l-2,l}(0)c_{l-2}. \end{aligned}$$

Example 5.2.2 *As a second example we consider the same dynamic equation, that is equation (5.1) on the time scale $\mathbb{T} = 2^{\mathbb{N}_0}$. Then $\sigma(t) = 2t$ and $\mu(t) = t$. The dynamic equation on $\mathbb{T} = 2^{\mathbb{N}_0}$ becomes*

$$y^{\Delta\Delta} - \frac{t^2}{2}y^{\Delta} + \lambda\left(1 + \frac{t^3}{2}\right)y = 0. \tag{5.7}$$

Let $\alpha = 1$. Using the fact that

$$h_0(t, 1) = 1,$$
$$h_1(t, 1) = \int_1^t \Delta\tau = t - 1,$$
$$h_2(t, 1) = \int_1^t (\tau - 1)\Delta\tau = \frac{t^2 - 3t + 2}{3},$$
$$h_3(t, 1) = \int_1^t \frac{\tau^2 - 3\tau + 2}{3}\Delta\tau = \frac{t^3 - 7t^2 + 14t - 8}{21},$$

we compute

$$-\frac{t^2}{2} = a_0 + a_1(t - 1) + a_2\frac{t^2 - 3t + 2}{3},$$

and

$$1 + \frac{t^3}{2} = b_0 + b_1(t - 1) + b_2\frac{t^2 - 3t + 2}{3} + b_3\frac{t^3 - 7t^2 + 14t - 8}{21},$$

that is,$a_0 = -\frac{1}{2}, a_1 = a_2 = -\frac{3}{2}, b_0 = \frac{3}{2}, b_1 = \frac{7}{2}$ and $b_2 = b_3 = \frac{21}{2}$. The recurrence relation for this particular time scale becomes,

$$c_{l+2} = -\sum_{n=l}^{l+3}\sum_{m=n-l}^{3}(a_m c_{n-m+1} + \lambda b_m c_{n-m})D_{m,n-m,l}(1).$$

We compute

$$
\begin{aligned}
c_2 &= -(a_0c_1+\lambda b_0c_0)D_{0,0,0}(1),\\
c_3 &= -(a_0c_2+\lambda b_0c_1)D_{0,1,1}(1)-(a_1c_1+\lambda b_1c_0)D_{1,0,1}(1)-(a_1c_2+\lambda b_1c_1)D_{1,1,1}(1),\\
c_4 &= -(a_0c_3+\lambda b_0c_2)D_{0,2,2}(1)-(a_1c_2+\lambda b_1c_1)D_{1,1,2}(1)-(a_2c_1+\lambda b_2c_0)D_{2,2,2}(1)\\
&\quad -(a_1c_3+\lambda b_1c_2)D_{1,2,2}(1)-(a_2c_2+\lambda b_2c_1)D_{3,2,2}(1)-(a_2c_3+\lambda b_2c_2)D_{2,3,2}(1).
\end{aligned}
$$

and for $l=3,4,\ldots$ *we have*

$$
\begin{aligned}
c_{l+2} &= -(a_0c_{l+1}+\lambda b_0c_l)D_{0,l,l}(1)-(a_1c_l+\lambda b_1c_{l-1})D_{1,l-1,l}(1)\\
&- (a_2c_{l-1}+\lambda b_2c_{l-2})D_{2,l-2,l}(1)-\lambda b_3c_{l-3}D_{3,l-3,l}(1)\\
&- (a_1c_{l+1}+\lambda b_1c_l)D_{1,l,l}(1)-(a_2c_l+\lambda b_2c_{l-1})D_{2,l-1,l}(1)-\lambda b_3c_{l-2}D_{3,l-2,l}(1)\\
&- (a_2c_{l+1}+\lambda b_2c_l)D_{2,l,l}(1)-\lambda b_3c_{l-1}D_{3,l-1,l}(1)-\lambda b_3c_lD_{3,l,l}(1).
\end{aligned}
$$

On $\mathbb{T}=2^{\mathbb{N}_0}$*, we have*
$\mu(t)=t$, $\mu^{\Delta}(t)=1$, $\mu^{\sigma}(t)=\mu(2t)=2t$.
$D_{0,0,0}(1)=D_{1,0,1}(1)=D_{0,1,1}(1)=D_{2,0,2}(1)=D_{0,2,2}(1)=1$,
$D_{1,1,1}(1)=\mu(1)=1$,
$D_{1,1,2}(1)=2+\mu^{\Delta}(1)=3$,
$D_{2,1,2}(1)=D_{1,2,2}(1)=\mu(1)+\mu^{\sigma}(1)=3$,
$D_{2,2,2}(1)=\mu(1)\mu^{\sigma}(1)=2$.
Hence, we obtain

$$
\begin{aligned}
c_2 &= \frac{1}{2}c_1-\frac{3}{2}\lambda c_0,\\
c_3 &= 2c_2+\left(\frac{3}{2}-2\lambda\right)c_1-\frac{7\lambda}{2}c_0,\\
c_4 &= \frac{7}{2}c_3+\left(12-\frac{17\lambda}{2}\right)c_2+\left(\frac{3}{2}-63\lambda\right)c_1-\frac{21\lambda}{2}c_0.
\end{aligned}
$$

For $l = 3, 4, \ldots$ we conclude

$$\begin{aligned}
c_{l+2} &= \left[\frac{1}{2}D_{0,l,l}(1) + \frac{3}{2}D_{1,l,l}(1) + \frac{3}{2}D_{2,l,l}(1)\right]c_{l+1} \\
&+ \left[\frac{3}{2}D_{1,l-1,1}(1) + \frac{3}{2}D_{2,l-1,1}(1) + \frac{\lambda}{2}(-3D_{0,l,l}(1) - 7D_{1,l,l}(1) - 21D_{2,l,l}(1) - 21D_{3,l,l}(1))\right]c_l \\
&+ \left[\frac{3}{2}D_{2,l-2,l}(1) + \frac{\lambda}{2}(-7D_{1,l-1,l}(1) - 21D_{2,l-1,l}(1) - 21D_{3,l-1,l}(1))\right]c_{l-1}. \\
&- \frac{21\lambda}{2}(D_{2,l-2,l} + D_{3,l-2,l})c_{l-2} - \frac{21\lambda}{2}D_{3,l-3,l}c_{l-3}.
\end{aligned}$$

5.2.2 Example 2

Consider the dynamic equation of second order

$$[1 - (\sigma(t))^2]y^{\Delta\Delta}(t) - (\sigma(t) + t)y^{\Delta} + k(k+1)y = 0, \tag{5.8}$$

on some time scales $\mathbb{T}$, where k is a constant.
We assume that

$$\begin{aligned}
1 - (\sigma(t))^2 &= \sum_{m=0}^{\infty} a_m h_m(t, \alpha), \\
-(\sigma(t) + t) &= \sum_{m=0}^{\infty} b_m h_m(t, \alpha),
\end{aligned}$$

and let the solution $y(t)$ be an infinite series in the form

$$y(t) = \sum_{n=0}^{\infty} c_n h_n(t, \alpha).$$

Using this expansion the equation (5.8) becomes

$$\sum_{m=0}^{\infty} a_m h_m \sum_{n=0}^{\infty} c_{n+2} h_n + \sum_{m=0}^{\infty} b_m h_m \sum_{n=0}^{\infty} c_{n+1} h_n + k(k+1) \sum_{n=0}^{\infty} c_n h_n = 0,$$

which can be written as

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (a_m c_{n-m+2} + b_m c_{n-m+1}) h_m h_{n-m} + k(k+1) \sum_{n=0}^{\infty} c_n h_n = 0.$$

We rewrite the above equation as

$$\sum_{l=0}^{\infty}\sum_{n=l}^{\infty}\sum_{m=n-l}^{n}\left[(a_m c_{n-m+2} + b_m c_{n-m+1})D_{m,n-m,l}(\alpha) + k(k+1)c_l\right] h_l(t,\alpha) = 0,$$

which yields the recurrence relation

$$\sum_{n=l}^{\infty}\sum_{m=n-l}^{n}\left((a_m c_{n-m+2} + b_m c_{n-m+1})D_{m,n-m,l}(\alpha)\right) + k(k+1)c_l = 0, l = 0, 1, 2, ...$$

Example 5.2.3 *As a particular case, we consider the dynamic equation (5.8) on the time scale* $\mathbb{T} = a\mathbb{N}_0$. *Then we have* $\sigma(t) = t + a$ *and* $\mu(t) = a$. *The dynamic equation (5.8) takes form*

$$(1 - a^2 + 2at - t^2)y^{\Delta\Delta} - (2t + a)y^{\Delta} + k(k+1)y = 0,$$

where

$$\begin{aligned}
1 - (\sigma(t))^2 &= 1 - a^2 - 2at - t^2 \\
&= (a_0 h_0(t,0) + a_1 h_1(t,0) + a_2 h_2(t,0)) \\
&= a_0 + a_1 t + a_2\left(\frac{t^2 - at}{2}\right), \\
-t - \sigma(t) &= -2t - a \\
&= b_0 h_0(t,0) + b_1 h_1(t,0) = b_0 + b_1 t.
\end{aligned}$$

that is, $a_0 = 1 - a^2, a_1 = -3a, a_2 = -2, b_0 = -a,$ *and* $b_1 = -2$. *The recurrence relation for this example can be written as*

$$\sum_{n=l}^{l+2}\sum_{m=n-l}^{n}\left((a_m c_{n-m+2} + b_m c_{n-m+1})D_{m,n-m,l}(0)\right) + k(k+1)c_l = 0, l = 0, 1, 2, ... \quad (5.9)$$

and if $l = 0$ *we have*

$$(a_0 c_2 + b_0 c_1)D_{0,0,0}(0) + k(k+1)c_0 = 0,$$

if $l = 1$

$$(a_0 c_3 + b_0 c_2)D_{0,1,1}(0) + (a_1 c_2 + b_1 c_1)D_{1,0,1}(0) + (a_1 c_3 + b_1 c_2)D_{1,1,1}(0) + k(k+1)c_1 = 0,$$

and for $l = 2, 3, ...$

$$(a_0 c_{l+2} + b_0 c_{l+1})D_{0,l,l}(0) + (a_1 c_{l+1} + b_1 c_l)D_{1,l-1,l}(0)$$

$$+a_2c_lD_{2,l-2,l}(0)+(a_1c_{l+2}+b_1c_{l+1})D_{1,l,l}(0)$$
$$+a_2c_{l+1}D_{2,l-1,l}(0)+a_2c_{l+2}D_{2,l,l}(0)+k(k+1)c_l=0.$$

Therefore, for $a \neq 1$ we compute

$$\begin{aligned} c_2 &= \frac{a}{1-a^2}c_1-\frac{k(k+1)}{1-a^2}c_0, \\ c_3 &= \frac{6a}{1-4a^2}c_2-\frac{(k(k+1)-2)}{1-4a^2}c_1, \end{aligned}$$

and

$$\begin{aligned} c_{l+2} = &-\frac{3aD_{1,l-1,l}(0)-2D_{2,l-1,l}(0)-3aD_{0,l,l}(0)-2D_{1,l,l}(0)}{(1-a^2)D_{0,l,l}(0)-3aD_{1,l,l}(0)-2D_{2,l,l}(0)}c_{l+1} \\ &-\frac{k(k+1)-2D_{2,l-2,l}(0)-2D_{1,l-1,l}(0)}{(1-a^2)D_{0,l,l}(0)-3aD_{1,l,l}(0)-2D_{2,l,l}(0)}c_l, \end{aligned}$$

for $l = 2, 3, ...$

Example 5.2.4 *We next consider $\mathbb{T} = 2^{\mathbb{N}_0}$. On this time scale we have $\sigma(t) = 2t$, and $\mu(t) = t$. Then the equation (5.8) is in the form*

$$(1-4t^2)y^{\Delta\Delta}-3ty^{\Delta}+k(k+1)y=0,$$

where k is a constant. Taking $\alpha = 1$ we compute

$$\begin{aligned} 1-4t^2 &= (a_0h_0+a_1h_1+a_2h_2)=a_0+a_1(t-1)+a_2\left(\frac{t^2-3t+2}{3}\right), \\ -3t &= b_0h_0+b_1h_1=b_0+b_1(t-1) \end{aligned}$$

that is, $a_0 = -3, a_1 = -12, a_2 = -12, b_0 = b_1 = -3$. The recurrence relation for this example is

$$\sum_{n=l}^{l+2}\sum_{m=n-l}^{n}\big((a_mc_{n-m+2}+b_mc_{n-m+1})D_{m,n-m,l}(1)\big)+k(k+1)c_l=0, l=0,1,2,..... \quad (5.10)$$

Then if $l = 0$ we have

$$(a_0c_2+b_0c_1)D_{0,0,0}(1)+k(k+1)c_0=0.$$

and if $l = 1$

$$(a_0c_3+b_0c_2)D_{0,1,1}(1)+(a_1c_2+b_1c_1)D_{1,0,1}(1)+(a_1c_3+b_1c_2)D_{1,1,1}(1)+k(k+1)c_1=0,$$

In general, for $l = 2, 3, ...$

$$(a_0c_{l+2} + b_0c_{l+1})D_{0,l,l}(1) + (a_1c_{l+1} + b_1c_l)D_{1,l-1,l}(1)$$
$$+a_2c_lD_{2,l-2,l}(1) + (a_1c_{l+2} + b_1c_{l+1})D_{1,l,l}(1)$$
$$+a_2c_{l+1}D_{2,l-1,l}(1) + a_2c_{l+2}D_{2,l,l}(1) + k(k+1)c_l = 0.$$

Consequently, we compute

$$\begin{aligned} c_2 &= -\frac{3}{3}c_1 + \frac{k(k+1)}{3}c_0, \\ c_3 &= -\frac{18}{15}c_2 + \frac{k(k+1)-3}{15}c_1, \end{aligned}$$

and for $l = 2, 3, 4, ...$ we obtain

$$\begin{aligned} c_{l+2} = &-\frac{12D_{1,l-1,l}(1) + 12D_{2,l-1,l}(1) + 3aD_{0,l,l}(1) + 3D_{1,l,l}(1)}{3D_{0,l,l}(1) + 12D_{1,l,l}(1) + 12D_{2,l,l}(1)}c_{l+1} \\ &-\frac{k(k+1) - 12D_{2,l-2,l}(1) - 3D_{1,l-1,l}(1)}{3D_{0,l,l}(1) + 12D_{1,l,l}(1) + 12D_{2,l,l}(1)}c_l. \end{aligned}$$

5.2.3 Example 3

Finally, we apply the series solution method to the dynamic equation

$$[1 - t\mu(t)]y^{\Delta\Delta}(t) - ty^{\Delta} + \lambda y(t) = 0, t \in \mathbb{T}. \qquad (5.11)$$

For $\alpha \in \mathbb{T}$ we assume that

$$1 - t\mu(t) = \sum_{m=0}^{\infty} a_m h_m(t, \alpha)$$

and let the solution be a series of the form

$$y(t) = \sum_{n=0}^{\infty} c_n h_n(t, \alpha).$$

Clearly

$$-t = -\alpha h_0(t, \alpha) - h_1(t, \alpha),$$

that is, we have $b_0 = -\alpha, b_1 = -1$. Then the equation (5.11) becomes

$$\sum_{m=0}^{\infty} a_m h_m(t, \alpha) \sum_{n=0}^{\infty} c_{n+2}h_n(t, \alpha) + (-\alpha h_0(t, \alpha) - h_1(t, \alpha)) \sum_{n=0}^{\infty} c_{n+1}h_n(t, \alpha) + \lambda \sum_{n=0}^{\infty} c_n h_n(t, \alpha) = 0,$$

or upon combining the series

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=n-m}^{n} a_m c_{n-m+2} D_{m,n-m,l}(\alpha) - \alpha \sum_{n=0}^{\infty} c_{n+1} h_0(t,\alpha) h_n(t,\alpha)$$
$$- \sum_{n=0}^{\infty} c_{n+1} h_1(t,\alpha) h_n(t,\alpha) + \lambda \sum_{n=0}^{\infty} c_n h_n(t,\alpha) = 0.$$

We rewrite the above equation as

$$\sum_{l=0}^{\infty}\sum_{n=l}^{\infty}\sum_{m=n-l}^{n} a_m c_{n-m+2} D_{m,n-m,l}(\alpha) h_l(t,\alpha) - \alpha \sum_{l=0}^{\infty} c_{l+1} D_{0,l,l}(\alpha) h_l(t,\alpha)$$
$$- \sum_{l=0}^{\infty} (c_{l+1} D_{0,l,l}(\alpha) + c_l D_{1,l-1,l}(\alpha)) h_l(t,\alpha) + \lambda \sum_{l=0}^{\infty} h_l(t,a) = 0$$

so the recurrence relation becomes

$$\sum_{n=l}^{\infty}\sum_{m=n-l}^{n} a_m c_{n-m+2} D_{m,n-m,l} - c_{1+l}(\alpha D_{0,l,l}(\alpha) + D_{1,l,l}(\alpha)) + c_l(\lambda - D_{1,l-1,l}(\alpha)) = 0, l = 0, 1, 2, ...$$

Example 5.2.5 *Let* $\mathbb{T} = 2^{\mathbb{N}_0}$. *On this time scale the equation (5.11) takes form*

$$(1 - t^2) y^{\Delta\Delta} - t y^{\Delta} + \lambda y(t) = 0.$$

Taking $\alpha = 1$, *we compute*

$$1 - t^2 = -3h_1(t,1) - 3h_2(t,\alpha)$$

that is, $a_0 = 0, a_1 = -3, a_2 = -3$. *On the other hand* $b_0 = b_1 = -1$. *The recurrence relation in this case takes form*

$$\sum_{n=l}^{\infty}\sum_{m=n-l}^{n} a_m c_{n-m+2} D_{m,n-m,l}(1) - c_{1+l}(D_{0,l,l}(1) + D_{1,l,l}(1)) + c_l(\lambda - D_{1,l-1,l}(1)) = 0, l = 0, 1, 2, ... \tag{5.12}$$

Therefore, we obtain

$$a_0 c_2 D_{0,0,0}(1) + \lambda c_0 = 0$$

for $l = 0$ *and*

$$-(D_{0,1,1}(1) + D_{1,1,1}(1)) c_2 + (\lambda - D_{1,0,1}(1)) c_1 - 3D_{1,0,1}(1) c_2 - 3D_{1,1,1}(1) c_3 = 0$$

for $l = 1$.

In general, for $l = 2, 3, ...,$ *the recurrence relation can be written as*

$$-(D_{0,l,l}(1) + D_{1,l,l}(1)) c_{l+1} + (\lambda - D_{1,l-1,l}(1)) c_1 - 3D_{1,l-1,l}(1) c_{l+1} - 3D_{2,,l-2,l}(1) c_l$$

$$-3D_{1,l,l}(1)c_{l+2} - 3D_{2,l-1,l}(1)c_{l+1} - 3D_{2,l,l}(1)c_{l+2} = 0.$$

As a result, for arbitrary c_0 *and* c_1 *we obtain*

$$\begin{aligned} c_2 &= \lambda c_0, \\ c_3 &= -\frac{5}{3}c_2 + \frac{\lambda - 1}{3}c_1, \end{aligned}$$

and

$$c_{l+2} = -\frac{D_{0,l,l}(1) + D_{1,l,l}(1) + 3D_{1,l-1,l}(1) + 3D_{2,l-2,l}(1)}{3D_{1,l,l}(1) + 3D_{2,l,l}(1)}c_{l+1} + \frac{\lambda - D_{1,l-1,l}(1) - 3D_{2,l-2,l}(1)}{3D_{1,l,l}(1) + 3D_{2,l,l}(1)}c_l,$$

for $l = 2, 3, ...$

CHAPTER 6

CONCLUSION

The series solution method discussed in this study is an analog of the series solution method for differential equations. The main idea is using Taylor series expansion about some point $\alpha \in \mathbb{T}$. In fact, this method covers the series solution method for ordinary differential equations since the set $\mathbb{R}$ of real numbers is a particular case of a time scale.

The main difficulty of this method is the computation of the coefficients $D_{n,m,l}(\alpha)$. In our study we derived some properties of these constants, like symmetry in the first two subscripts. However, as m, n, l become large, even greater than 3, the computation becomes very complicated. The examples of nonconstant coefficients solved in Chapter 5 show clearly that it is very difficult to derive a general expression for the coefficients of the dependent variable $y(t)$ even in simple cases such as dynamic equations with polynomial coefficients. The main reason for this difficulty is the presence of the graininess function $\mu(t)$, which is 0 if $\mathbb{T} = \mathbb{R}$ and hence, the constants $D_{n,m,l}(\alpha)$ are simplified considerably on this time scale. Otherwise, if $\mu(t) \not\equiv 0$ and moreover if $\mu(t)$ is nonconstant the calculation of the constants $D_{n,m,l}(\alpha)$ becomes very complicated.

REFERENCES

[1] R. P. Agarwal and M. Bohner, *Basic Calculus on Time Scales and some of its Applications*, Result.Math. 35(1999) 3-22.

[2] R. P. Agarwal, M. Bohner and P.J.Y. Wong, *Sturm Liouville eigenvalue problems on time scales*, Appl. Math.Comput. 99(1999) 153-166.

[3] R. P. Agarwal, and D. O'Regan, *Triple solutions to boundary value problems on time scales*, Appl. Math. Lett 13(4) (2000) 7-11.

[4] R. P. Agarwal, and D. O'Regan, *Nonlinear boundary value problems on time scales*, Nonlinear Anal. Appl 250(2000) 561-578.

[5] R. P. Agarwal, M. Bohner, D. O'Regan and A. Peterson, *Dynamic equations on time scales: a survey. Dynamic equations on time scales*, J. Comput. Appl. Math. 141 (2002) 1-26.

[6] R. P. Agarwal, M. Benchohra, D. O'Regan and A. Ouahab, *Second order impulsive dynamic equations on time scales*, Funct. Differ. Equ., **11** (2004) 223-234.

[7] E. Akin, *Boundary value problems for a differential equation on a measure chain,Panamar*, Math. J. 10(3)(2000) 17-30.

[8] F.M Atici, G.Sh. Guseinov, and B. Kaymakcalan, *Stability criteria for dynamic equations on time scales with periodic cofficients*. In: Conference Proceedings of the Third International Conference on Dynamic Systems and Applications, Atlanta, 2001.

[9] B. Aulbach and S. Hilger, *Linear dynamic processes with inhomogeneous time scales*, In Nolinear Dynamics and Quantum Dynamical Systems(Gaussig 1990), Vol. 59 of Math. Res. Academie Verlag, Berlin, (1990), 9-20.

[10] B. Aulbach and S. Hilger, *A unified approach to continuous and discrete dynamics, Qualitative Theory of Differential Equations, Szeged, 1988,Colloq*, Math. Soc. Janos Bolyai, vol. 53, North-Holland, Amsterdam, (1990),pp 37-56.

[11] M. Bohner, *Calculus of variations on time scales*.2000. Dynamic systems and applications, vol.13(2004)339-349.

[12] M. Bohner, O. Dosly, and R. Hilscher, *Linear Hamiltonian dynamic systems on time scales: Sturmian property of the principal solution*. Nonlinear Anal., (2001). To appear.

[13] M. Bohner, A. Peterson, *A survey of exponential functions on time scales*. Rev. Cubo Mat. Educ., 3(2) (2001), 285-301.

[14] M. Bohner, A. Peterson, *First and second order linear dynamic equations on measure chains*. J. Differ. Equations Appl., 7(6) (2001), 767-792.

[15] M. Bohner, *Discrete Linear Hamiltonian eigenvalue problems*, Compute. Math. Appl. 36(10-12)(1998) 179-192.

[16] M. Bohner and A. Peterson, *Dynamic Equation on Time Scales*, An Introduction with Applications. Birkhauser, Boston, (2001).

[17] M. Bohner and A. Peterson, *Advances in Dynamic Equation on Time Scales*. Birkhauser, Boston, (2003).

[18] M. Bohner, G.SH. Guseinov and A. Peterson, *Introduction to the time scales calculus, in Advances in Dynamic Equations on Time Scales (Birkhauser Boston,Boston,MA,2003)*,pp 1-15.

[19] M. Bohner and G.SH. Guseinov, *Partial differentiation on time scales*. Dyn.Syst.Appl.13(3-4),351-379(2004).

[20] M. Bohner and G.SH. Guseinov, *Multiple integration on time scales*. Dyn.Syst.Appl.14(3-4),579-606(2005).

[21] M. Bohner and G.SH. Guseinov, *Linear integrals and Green's formula on time scales*. J.Math..Anal.Appl.326(2),1124-1141(2007).

[22] M. Bohner and S.G. Georgiev, *Multivariable Dynamic Calculus on Time Scales*. Sprnger, (2016)

[23] L. Erbe, A. Peterson, *Green's functions and comparison theorems for differential equations on measure chains,Dynam*, Contin. Discrete Impuls. Systems 6(1)(1999) 121-137.

[24] L. Erbe, A. Peterson, *Riccati equations on a measure chain*, Dynam. Systems Appl. (2001) 6(1)(1999) 121-137.

[25] L. Erbe, A. Peterson, *Oscillation criteria for second order matrix dynamic equations on a time scale, this issue, J.*, Comput. Appl. Math. 141(2002) 169-185.

[26] S. Georgiev, *Integral Equations on Time Scales*. Atlantis Press, (2016).

[27] B. Haile and L. Hall, *Polynomial and series solutions of dynamic equations on time scales*. Dynamic Systems and Appl., 11(2002).

[28] S. Hilger, *Ein Mabkettenkalkul mit Anwendung auf Zentrumsmannigfaltigkeiten*. PhD thesis, Universitat Wurzburg, (1988).

[29] S. Hilger, *Analysis on measure chains-a unified approach to continous and discrete calculus*. Results Math., (1990),18:18-56.

[30] S. Hilger, *Differential and difference calculus-unified!*, Nonlinear Anal. 30(5)(1997) 2683-2694.

[31] S. Hilger, *Special functions, Laplace and Fourier transform on measure chains, Dynam*, System Appl. 8(3-4)(1999) 471-488(Special Issue on "Discrete and Continuous Hamiltonian Systems", edited by R.P Agarwal and M.Bohner).

[32] J. Hoffacker, *Basic partial dynamic equations on time scales*. J.Difference Equ. Appl., (2002),8(4):307-319.

[33] B. Kaymakçalan, *A survey of dynamic systems on measure chains*, Funct. Differ. Equ. 6(1-2)(1999) 125-135.

[34] B. Kaymakçalan, V. Lakshmikantham, and S. Sivasundaraem, *Dynamic Systems on Measure Chains*, vol. 370 of Mathematics and its Applications. Kluwer Academic Publishers Group,Dordrecht,1996.

[35] Hsuan-Ku Liu, *Developing a series solution method of q-difference equation*. J.Appl. Math.,2013 Art.Id.743973.

[36] D. Mozyrska and E. Pawluszewicz, *Hermite's equations on time scales*. Appl. Math. Lett.,22(2009) 1217-1219.

[37] Z. Pospisil, *Hyperbolic sine and cosine functions on measure chains*. Nonlinear Anal., (2001).