NEW PRECONDITIONERS FOR STATIONARY ITERATIVE METHODS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
ATILIM UNIVERSITY

BY

NAIMA IBRAHIM ATYA

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS

JULY 2017

Approval of the Graduate School of Natural and Applied Sciences, Atılım University.

Prof. Dr. Ali Kara
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of **Master of Science in Mathematics Department, Atılım University**.

Prof. Dr. Tanıl Ergenç
Head of Department

This is to certify that we have read the thesis NEW PRECONDITIONERS FOR STATIONARY ITERATIVE METHODS submitted by NAIMA IBRAHIM ATYA and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Prof. Dr. Ahmet Yaşar Özban
Supervisor

Examining Committee Members:

Prof. Dr. Bülent Karasözen
Mathematics Department, Atilim University

Prof. Dr. Ahmet Yaşar Özban
Mathematics Department, Atilim University

Assoc. Prof. Dr. Mehmet TURAN
Mathematics Department, METU

Date: July 14, 2017

I declare and guarantee that all data, knowledge and information in this document has been obtained, processed and presented in accordance with academic rules and ethical conduct. Based on these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name : NAIMA IBRAHIM ATYA

Signature :

ABSTRACT

NEW PRECONDITIONERS FOR STATIONARY ITERATIVE METHODS

Atya, Naima Ibrahim

M.S., Department of Mathematics

Supervisor : Prof. Dr. Ahmet Yaşar Özban

July 2017, 63 pages

The convergence of an iterative method used for the solution of systems of linear equation depends on the properties of the spectrum of the matrix of the linear system. So, to speed up the convergence, the given linear system is transformed into an equivalent one by linear transformations, known as preconditioners.

In this thesis, we introduce two new preconditioners for Jacobi and Gauss-Seidel (GS) iterative methods for the solution of linear systems with strictly columnwise diagonally dominant (SCDD) $L-$ matrices and SCDD positive matrices. The new preconditioners can be applied on a single row, on a limited number of rows, called partial preconditioning, or on all rows, called complete preconditioning, of the system matrix. First of all, the properties of the preconditioned matrices are determined for systems with SCDD $L-$ matrices and SCDD positive matrices. Then convergence analysis of the Jacobi and GS iterative methods are performed for the preconditioned systems.

For systems with SCDD $L-$ matrices, it is shown that the spectral radii of Jacobi and GS iteration matrices for preconditioned systems are smaller than the ones associated with unpreconditioned systems. Although, for systems with SCDD positive matrices, we prove that the spectral radii of Jacobi iteration matrices for preconditioned systems are smaller than the ones associated with unpreconditioned systems, no such result is available for GS iteration matrices.

Numerical results show that for systems with SCDD $L-$matrices, the new preconditioners are quite competitive with the ones existing in the literature in the sense of spectral radii and the number of iterations. Nevertheless, for systems with SCDD positive matrices, numerical results demonstrate that although new preconditioners are still competitive with some other preconditioners, usually they are not preferable on many of the already existing ones. Finally, the performances of new preconditioners for CDD $L-$matrices and non-CDD $L-$matrices and even for non-SCDD positive matrices deserve further research.

Keywords: Systems of linear equations, Preconditioning, Preconditioners, Iterative methods, Convergence

ÖZ

DURAĞAN YİNELEMELİ YÖNTEMLER İÇİN YENİ ÖN KOŞULLAYICILAR

Atya, Naima Ibrahim

Yüksek Lisans, Matematik Bölümü

Tez Yöneticisi : Prof. Dr. Ahmet Yaşar Özban

Temmuz 2017, 63 sayfa

Doğrusal denklem sistemlerinin çözümü için kullanılan yinelemeli yöntemlerin yakınsaklığı doğrusal sistem matrisinin spektrumunun özelliklerine bağlıdır. Bu nedenle, yakınsaklığı hızlandırmak için, verilen doğrusal sistem, ön koşullayıcılar olarak bilinen doğrusal dönüşümlerle eşdeğer bir sisteme dönüştürülür.

Bu tezde, sütuna göre kesin köşegensel baskın (SKKB) $L-$matrisli ve SKKB pozitif matrisli doğrusal denklem sistemlerinin Jacobi ve Gauss-Seidel (GS) yöntemleriyle çözümü için iki yeni ön koşullayıcı tanımlanmaktadır. Yeni ön koşullayıcılar sistem matrisinin tek bir satırına, sınırlı sayıdaki satırlarına, ki kısmi ön koşullama olarak adlandırılır, veya bütün satırlarına, ki tam ön koşullama olarak adlandırılır, uygulanabilir. İlk olarak, SKKB $L-$matrisler ve SKKB pozitif matrisler için ön koşullanmış matrislerin özellikleri belirlenmektedir. Daha sonra ön koşullandırılmış sistemler için Jacobi ve GS yöntemlerinin yakınsaklık analizleri yapılmaktadır.

SKKB $L-$matrisli sistemler için, ön koşullandırılmış sistemlerin Jacobi ve GS yineleme matrislerinin spektral yarıçaplarının ön koşullandırılmamış sistemlere karşılık gelenlerden daha küçük olduğu gösterilmektedir. SKKB pozitif matrisli sistemler için, ön koşullandırılmış sistemlerin Jacobi yineleme matrislerinin spektral yarıçaplarının

ön koşullandırılmamış sistemlere karşılık gelenlerden daha küçük olduğunu ispatlamamıza karşın, GS yineleme matrisleri için böyle bir sonuç mevcut değildir.

Sayısal sonuçlar, yeni ön koşullayıcıların, SKKB $L-$matrisli sistemler için spektral yarıçap ve yineleme sayısı bakımından literatürde mevcut olanlarla tümüyle yarışabilir durumda olduğunu göstermektedir. Buna karşılık, SKKB pozitif matrisli sistemlere ilişkin sayısal sonuçlar, yeni ön koşullayıcıların diğer bazı önkoşullayıcılarla yarışabilir nitelikte olmasına karşın, mevcut ön koşullayıcıların çoğuna karşı tercih edilebilir olmadığını ifade etmektedir. Son olarak, yeni ön koşullayıcıların sütuna göre köşegensel baskın (SKB) $L-$matrisler ve SKB-olmayan $L-$matrisler ve hatta, SKKB-olmayan pozitif matrisler için etkinlikleri, daha fazla araştırmayı haketmektedir.

Anahtar Kelimeler: Doğrusal denklem sistemleri, Ön koşullama, Ön koşullayıcılar, Yinelemeli yöntemler, Yakınsaklık.

ACKNOWLEDGMENTS

First of all, I would like to thank Allah, the merciful and the compassionate, for everything given to me.

I would like to thank my country which gave me this dispatch and helped me to end it by the scholarship.

Also, I would like to thank my supervisor Prof. Dr. Ahmet Yaşar Özban for his continuous assistance during the thesis.

Finally I would like to thank my husband, my children and my parents for helping me and praying me all the time and, I pray Allah to heal my sweet daughter Saja.

TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF SYMBOLS

$\mathbb{R}$	:	the set of real numbers
$\mathbb{R}^n$	:	the n-dimensional real space (real column vectors)
$\mathbb{C}^n$	:	the n-dimensional complex space (complex column vectors)
$\mathbb{R}^{m\times n}$	:	the set of all $m \times n$ real matrices
$\mathcal{M}_n$	:	the set of all $n \times n$ real matrices
$\mathbb{R}_0^+$	:	the set of all nonnegative real numbers

CHAPTER 1

INTRODUCTION

To enter the topic as briefly as possible, we quote two famous numerical mathematicians. *"Numerical linear algebra is an exciting field of research and much of this research has been triggered by a problem that can be posed simply as: given* $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^{m}$*, find solution vector(s)* $x \in \mathbb{C}^{n}$ *such that* $Ax = b$*. Many scientific problems lead to the requirement to solve linear systems of equations as part of the computations"* [45]. As quite well-known examples, linear systems arise in the numerical solution of partial differential equations by different methods, such as finite differences, finite elements and boundary elements and, in the numerical solution of boundary value problems.

Mainly, there are two types of methods: *Direct methods* which generally relies on Gaussian elimination and *Iterative methods* which produce a sequence of vectors that converges to the solution. Direct methods are preferred, in general, for dense linear systems, in particular, with multiple right hand sides. For a nonsingular system where $m = n$, Gaussian elimination requires, with all its enhancements to solve *instability* problems, $\mathcal{O}(n^3)$ additions and multiplications [45]. Iterative methods, on the other hand, are preferred for sparse linear systems, that is linear systems which contains *very few* non-zero elements. But there is no precise boundary. Iterative methods can be preferred for dense linear systems, due to the type of the problem under consideration, such as in [46] or due to the problem and the computer environment in which the solution is searched, such as in [33].

In the literature, there are a lot of research articles on *iterative methods* and preconditioning. For a great majority of the preconditioners for stationary iterative methods, such as *Jacobi, Gauss-Seidel* (GS), *Successive Overrelaxation* (SOR), *Acceler-*

ated Overrelaxation (AOR) etc., have their roots in the one introduced by Mokari-Bolhassan and Trick [36]. However, its effect became realizable after its much more mathematical and rigorous treatment in [35]. Then Gunawardena et al., [8] lead the way. The articles [3], [13], [16]-[25], [28], [29]-[32], [34], [37]-[44], [47], [51]-[54], [56], [58] and [60]-[62] are some of the articles all about preconditioners for Jacobi and Gauss-Seidel iterative methods, a great majority of them being devoted to Gauss-Seidel method. On the other hand, [6], [11], [12], [26], [27], [55], [57] and [59] are some of the articles which introduces and discusses new preconditioners for AOR or SOR methods.

The structure of the thesis is as follows. Chapter 2 is devoted to definitions, concepts and some known results from the literature about numerical solution of systems of linear equations by stationary iterative methods and preconditioning. In this context, convergence of the vector and matrix sequences, the properties and the importance of the spectral radius of stationary iterative methods and the effect of preconditioning on the spectral radius of iteration matrices are studied and emphasized. Finally, some preconditioners existing in the literature have been given.

In Chapter 3, for the iterative solution of systems of linear equations by Jacobi and GS iterative methods, mainly two types of new preconditioners have been introduced. The new preconditioners can be applied on a single row, on some number of rows or on the whole coefficient matrix. After investigating the properties of the preconditioned coefficient matrices, the convergence analysis of the Jacobi and GS iterations have been carried out and some inequalities between the spectral radii of the iteration matrices associated with the preconditioned and unpreconditioned coefficient matrices are obtained.

Chapter 4 covers some numerical test results on some matrices and linear systems of different orders for some of the preconditioners in the literature and for the newly developed preconditioners. In this sense, spectral radii of Jacobi and GS iteration matrices for unpreconditioned and preconditioned system matrices have been given and compared. Moreover, some linear systems and associated preconditioned linear systems have been solved by Jacobi and GS methods to observe the efficiency of the new preconditioners and the effect of preconditioning on number of iterations.

CHAPTER 2

PRELIMINARY DEFINITIONS, CONCEPTS AND RESULTS

2.1 Solution of Linear Systems and Iterative Methods

Given a linear system

$$Ax = b, \tag{2.1}$$

where $A \in \mathbb{R}^{m\times n}$ is the *coefficient matrix* of the system, $x \in \mathbb{R}^n$ is the *vector of unknowns* and $b \in \mathbb{R}^m$ is the *vector of known values*, the problem is to find a vector, say, $x^* \in \mathbb{R}^n$ such that $Ax^* = b$ or, equivalently, $Ax^* - b = \mathbf{0} \in \mathbb{R}^n$. Depending on the structure and the properties of A and b the linear system may have a unique solution, infinitely many solutions or no solution. Throughout the thesis it will be assumed that $m = n$, i.e., A is a square matrix, and the system has a unique solution. It is a well known fact from the linear algebra that, if A is nonsingular, i.e., $\det(A) \neq 0$, then the linear system has a unique solution. We shall use the notation $\mathcal{M}_n$ to denote the set of all real $n \times n$ matrices and x^* will be used to represent the unique solution of the linear system.

For the solution of linear systems of the form (2.1), we mainly have two types of methods:

(a) Direct Solution Methods, such as *Gaussian Elimination, LU-Decomposition, Cholesky Factorization*. These methods find the exact solution $x^* \in \mathbb{R}^n$, apart from the rounding errors, in a finite number of steps.

(b) Iterative Methods, like *Jacobi*, *Gauss-Seidel* (GS), *Successive Overrelaxation* (SOR), *Accelerated Overrelaxation* (AOR) and *Symmetric Successive Overre-*

laxation (SSOR), which are known as *stationary iterative methods*; and, *nonstationary iterative methods*, such as *Conjugate Gradient* (CG), *Minimum Residual* (MINRES), *Generalized Minimal Residual* (GMRES), *BiConjugate Gradient* (BiCG) and *Chebyshev Method*.

For a detailed information about iterative methods for the solution of linear systems, one may refer to *Numerical Linear Algebra* by L.N.Trefethen&D.Bauu, III [50].

To find the solution of (2.1), iterative methods usually start with an initial approximation $x^{(0)} \in \mathbb{R}^n$ to x^* and compute iteratively a sequence $\left\{x^{(k)}\right\}_{k=0}^{\infty}$ of approximations to x^*, without ever getting x^*, in general.

The general form of stationary iterative methods, such as *Jacobi*, GS, SOR, AOR and SSOR for the linear system (2.1) can be given as

$$x^{(k+1)} = Tx^{(k)} + c, \quad k = 0, 1, 2, \ldots,$$

where $x^{(0)} \in \mathbb{R}^n$ is the initial approximation to the solution x^* of (2.1), T is the *iteration matrix* constructed using the coefficient matrix A, and $c \in \mathbb{R}^n$ is the vector constructed using A and the vector b. Before going further, it will be useful to introduce some concepts and results.

2.2 Vector and Matrix Norms

Let $\mathbb{R}^n$ denote the set of all real $n-$ dimensional column vectors. To define the size or length of a vector in $\mathbb{R}^n$ we use vector *norm*s.

Definition 2.2.1 *([2], pp.432) A vector norm on $\mathbb{R}^n$ is a function, $\|.\|$, from $\mathbb{R}^n$ into $\mathbb{R}_0^+$ with the following properties :*

(a) $\|x\| \geq 0$ *for all* $x \in \mathbb{R}^n$.

(b) $\|x\| = 0$ *if and only if* $x = 0$.

(c) $\|\alpha x\| = |\alpha| \, \|x\|$ *for all* $\alpha \in \mathbb{R}$ *and* $x \in \mathbb{R}^n$.

(d) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in \mathbb{R}^n$.

Definition 2.2.2 *([10], pp.264-265). The sum norm, the Euclidean norm and the max norm for the vector* $x = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$ *are defined by :*

$$\|x\|_1 = |x_1| + \cdots + |x_n|, \; \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \quad \textit{and} \quad \|x\|_\infty = \max\{|x_1|, \dots, |x_n|\},$$

respectively.

Similar definition can be given on $\mathbb{C}^n$, the set of all complex $n-$ dimensional column vectors. The *distance* between any two vectors x and y in $\mathbb{R}^n$ is defined by $\|x - y\|$, where $\|\cdot\|$ is any vector norm.

In the solution of linear systems by iterative methods, we use vector norms to measure the distance between the *exact* and *approximate* solutions, and the most important problem is whether the sequence $\left\{x^{(k)}\right\}_{k=0}^{\infty}$ of approximation vectors in $\mathbb{R}^n$ *converges* to the unique solution, say, $x^* \in \mathbb{R}^n$ of a given linear system $Ax = b$.

Definition 2.2.3 *([2], pp.435). A sequence* $\left\{x^{(k)}\right\}_{k=0}^{\infty}$ *of vectors in* $\mathbb{R}^n$ *is said to converge to a vector* $x^* \in \mathbb{R}^n$ *with respect to the vector norm* $\|.\|$*, if, given any* $\varepsilon > 0$*, there exists an integer* $N(\varepsilon)$ *such that*

$$\left\|x^{(k)} - x^*\right\| < \varepsilon, \quad \textit{for all} \quad k \geq N(\varepsilon).$$

Definition 2.2.4 *([2], pp.436). The sequence of vectors* $\left\{x^{(k)}\right\}_{k=0}^{\infty}$ *converges to* x^* *in* $\mathbb{R}^n$ *with respect to the vector norm* $\|\cdot\|_\infty$*, if and only if* $\lim_{k\to\infty} x_i^{(k)} = x_i^*$ *for each* $i = 1, 2, ..., n$.

In some cases, different vector norms can be used for different purposes. The following result establishes the relation between any two vector norms, say, $\|\cdot\|_{N_1}$ and $\|\cdot\|_{N_2}$.

Theorem 2.2.5 *([48], pp.177). All vector norms on* $\mathbb{R}^n$ *are equivalent in the sense that for each pair of norms* $\|\cdot\|_{N_1}$ *and* $\|\cdot\|_{N_2}$*, there exist constants* $m, M > 0$ *satisfying*

$$m\|x\|_{N_2} \leq \|x\|_{N_1} \leq M\|x\|_{N_2} \quad \textit{for all } x \in \mathbb{R}^n.$$

The analysis of some methods and algorithms involving matrices or solution of linear systems by direct methods in general and iterative methods in particular, requires matrix norms. For example, the convergence of a *stationary iterative method* for the solution of linear systems depends on the *spectral radius* of the iteration matrix. Hence, estimates for the spectral radius or *sensitivity* of solutions to small perturbations in the linear systems are given in terms of *matrix norms*, in general.

Definition 2.2.6 *([7], pp.9). A function* $||.|| : A \in \mathbb{R}^{m\times n} \to \mathbb{R}_0^+$ *is called a matrix norm if for all* $A, B \in \mathbb{R}^{m\times n}$ *and all scalars c, the following conditions are satisfied.*

(a) $||A|| \geq 0$ *and* $||A|| = 0$ *if and only if* $A = \mathbf{0}$.

(b) $||cA|| = |c|\,||A||$.

(c) $||A + B|| \leq ||A|| + ||B||$.

(d) $||AB|| \leq ||A|| \cdot ||B||$.

Let $A \in \mathbb{R}^{m\times n}$. Some commonly used matrix norms are:

(a) The *maximum column sum matrix norm* $||A||_1$ of A, is defined by

$$||A||_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} \left|a_{ij}\right|. \tag{2.2}$$

(b) The *maximum row sum matrix norm* $||A||_\infty$ of A, is defined by

$$||A||_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} \left|a_{ij}\right|. \tag{2.3}$$

(c) The 2*−norm* or *Euclidean norm* $||A||_2$ of A, is defined by

$$||A||_2 = \left[\rho\left(A^T A\right)\right]^{\frac{1}{2}} = \left[\rho\left(AA^T\right)\right]^{\frac{1}{2}}. \tag{2.4}$$

(d) The *Frobenius norm* $||A||_F$ of A, is defined by

$$||A||_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \left|a_{i,j}\right|^2\right)^{1/2}$$

Remark 2.2.7 *Although matrix norm definitions are given for $m \times n$ matrices, we shall use them for $m = n$, since our problem is to study the stationary iterative methods for the solution of $Ax = b$, where $A \in \mathcal{M}_n$.*

Having given the definitions and examples of some vector and matrix norms, we can state the following.

Theorem 2.2.8 *([2], pp.438). If $\|\cdot\|$ is a vector norm on $\mathbb{R}^n$ and $A \in \mathcal{M}_n$, then*

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

is a matrix norm.

A matrix norm may or may not be obtained from a vector norm. Matrix norms which can be obtained by vector norms are called *induced* or *natural* matrix norms associated with the vector norm. Throughout the thesis all matrix norms that are used are natural matrix norms. The matrix norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ defined by (2.2), (2.4) and (2.3) are the *natural* or *induced* matrix norms associated with the vector norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ give in Definition (2.2.2).

Theorem 2.2.9 *([2], pp.438). For any vector $z \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, matrix $A \in \mathcal{M}_n$, and any natural matrix norm $\|.\|$,*

$$\|Az\| \leq \|A\| \, \|z\| \, .$$

Similar to the distance between vectors in $\mathbb{R}^n$, the distance between the two matrices $A \in \mathcal{M}_n$ and $B \in \mathcal{M}_n$ with respect to the matrix norm under consideration is $\|A - B\|$.

Although matrix norms can be obtained in various ways, the norms considered most frequently are the ones induced by the vector norms $\|\cdot\|_2$ and $\|x\|_\infty$:

$$\|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty$$

and

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 \, .$$

Definition 2.2.10 *([2], pp.442). Let $A \in \mathcal{M}_n$. If λ is a scalar (real or complex number) and $x \in \mathbb{R}^n$ (or $x \in \mathbb{C}^n$) is a nonzero vector such that*

$$Ax = \lambda x,$$

then λ is an eigenvalue or characteristic value of A and $x \in \mathbb{C}^n \backslash \{\mathbf{0}\}$ is an eigenvector associated with the eigenvalue λ.

Definition 2.2.11 *([2], pp.443). If $A \in \mathcal{M}_n$, the characteristic polynomial of A is defined by*

$$p_A(\lambda) = \det(\lambda I - A).$$

Definition 2.2.12 *([2], pp.443). If $p_A(\lambda)$ is the characteristic polynomial of the matrix $A \in \mathcal{M}_n$, the zeros of $p_A(\lambda)$ are the eigenvalues, or the characteristic values, of A.*

Theorem 2.2.13 *([4], pp.375). The eigenvalues of a triangular matrix are its diagonal entries.*

Definition 2.2.14 *([4], pp.375). The set $\sigma(A)$ of all eigenvalues of a matrix $A \in \mathcal{M}_n$ is called the spectrum of A.*

Theorem 2.2.15 *([10], pp.42). If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A \in \mathcal{M}_n$, then $\det(A) = \prod_{i=1}^{n} \lambda_i$.*

Definition 2.2.16 *([2], pp.446). The spectral radius $\rho(A)$ of a matrix $A \in \mathcal{M}_n$ is defined by*

$$\rho(A) = \max_{1 \leq i \leq n} |\lambda_i|$$

where $\lambda_i \in \sigma(A)$.

Given $A \in \mathcal{M}_n$, finding $\sigma(A)$ and, hence, finding $\rho(A)$ is a quite challenging problem. But in many problems, instead of the spectral radius, an upper bound for it is sufficient. In this sense, natural matrix norms play a crucial role and supply upper bounds for the spectral radius of a matrix $A \in \mathcal{M}_n$ as given in the following result.

Theorem 2.2.17 *([4], pp.30). Let λ be an eigenvalue of a matrix $A \in \mathcal{M}_n$. Then for any natural matrix norm*

$$|\lambda| \leq \|A\| .$$

Proof. *If λ is any eigenvalue of A and x is an eigenvector associated with the eigenvalue λ, then $Ax = \lambda x$. Thus*

$$|\lambda| . \|x\| = \|\lambda x\| = \|Ax\| \leq \|A\| \|x\| .$$

Dividing both sides by $\|x\| > 0$, we get $|\lambda| \leq \|A\|$. □

Corollary 2.2.18 *([2], pp.446). Let $A \in \mathcal{M}_n$. Then for any natural matrix norm $\|\cdot\|$,*

$$\rho(A) \leq \|A\| .$$

In studying iterative matrix techniques, it is quite important to know when powers of a matrix become small (that is, when all the entries approach zero). Matrices of this type are called *convergent*.

Definition 2.2.19 *([2], pp.448). A matrix $A \in \mathcal{M}_n$ is called convergent if $A^k \to \mathbf{0}$, as $k \to \infty$, that is $\lim_{k\to\infty} A^k = \mathbf{0}$.*

Theorem 2.2.20 *([2], pp.449). Let $A \in \mathcal{M}_n$. The following statements are equivalent.*

(a) A is a convergent matrix.

(b) $\lim_{n\to\infty} \|A^n\| = 0$ for some natural matrix norm.

(c) $\lim_{n\to\infty} \|A^n\| = 0$ for all natural matrix norms.

(d) $\rho(A) < 1$.

(e) $\lim_{n\to\infty} A^n x = 0$ for every $x \in \mathbb{R}^n$.

Theorem 2.2.21 *([4], pp.31). Let $A \in \mathcal{M}_n$. The infinite series of matrices*

$$I + A + A^2 + \cdots$$

converges to $(I - A)^{-1}$ if A is a convergent matrix.

Proof. If A is a convergent matrix, then we must have $|\lambda_i| < 1$ for each eigenvalue λ_i of A. Since the eigenvalues of $(I - A)$ are $1 - \lambda_1, 1 - \lambda_2, \ldots, 1 - \lambda_n$, $|\lambda_i| < 1$ implies that none of the eigenvalues of $(I - A)$ is zero. Now, from identity,

$$(I - A)(I + A + A^2 + \ldots + A^k) = I - A^{k+1},$$

we have,

$$I + A + A^2 + \cdots + A^k = (I - A)^{-1} - (I - A)^{-1} A^{k+1}.$$

Since $A^k \to 0$, as $k \to \infty$, we get $I + A + A^2 + \cdots + A^k + \cdots = (I - A)^{-1}$. □

Theorem 2.2.22 *([4], pp.29). The sequence $\left\{x^{(k)}\right\}_{k=0}^{\infty}$ of vectors in $\mathbb{R}^n$ converges to the vector $x^* \in \mathbb{R}^n$ if and only if*

$$\lim_{k\to\infty} \left\|x^{(k)} - x^*\right\| = 0,$$

where $\|\cdot\|$ is any vector norm.

Theorem 2.2.23 *([4], pp.30). The sequence of matrices $A^{(1)}, A^{(2)}, \ldots,$ converges to the matrix A if and only if*

$$\lim_{k\to\infty} \left\|A - A^{(k)}\right\| = 0,$$

where $\|\cdot\|$ is any matrix norm.

2.3 Stationary Iterative Methods

In constructing stationary iterative methods for the solution of the linear system $Ax = b$, the basic idea is to split the coefficient matrix $A \in \mathcal{M}_n$ as

$$A = M - N,$$

where M is nonsingular. So, the linear system $Ax = b$ is equivalent to the linear system $(M - N)x = b$ or $Mx = Nx + b$ or

$$x = M^{-1}Nx + M^{-1}b$$

Then, given $x^{(0)} \in \mathbb{R}^n$, equation (2.3) suggests the iterative method

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, \quad k = 0, 1, \ldots,$$

or

$$x^{(k+1)} = Tx^{(k)} + c, \quad k = 0, 1, \ldots,$$

where $T = M^{-1}N$ is the *iteration matrix*, $c = M^{-1}b$ and $x^{(0)}$ is the initial approximation to the solution. It is clear that different choices of M and N lead to different *stationary iterative methods*. The name *stationary iterative method* refers to the fact that the iteration matrix T does not change during the iteration.

The coefficient matrix $A \in \mathcal{M}_n$ of the linear system (2.1) can be written as

$$A = D - L - U,$$

where D, $-L$ and $-U$ are the *diagonal*, *strictly lower triangular* and *strictly upper triangular parts* of A, respectively. Assume that D is nonsingular. Then, letting $M = D$ and $N = L + U$ we get

$$Dx = (L + U)x + b \tag{2.5}$$

or, since M is nonsingular,

$$x = D^{-1}(L + U)x + D^{-1}b,$$

which suggests the well-known *Jacobi iteration method*:

$$x^{(k+1)} = D^{-1}(L + U)x^{(k)} + D^{-1}b, \quad k = 0, 1, \ldots.$$

Here $T_J = D^{-1}(L + U)$ is the *Jacobi iteration matrix*. If $M = D - L$ and $N = U$, then we get

$$(D - L)x = Ux + b,$$

or

$$x = (D - L)^{-1}Ux + (D - L)^{-1}b,$$

which leads to the method

$$x^{(k+1)} = (D - L)^{-1}Ux^{(k)} + (D - L)^{-1}b, \quad k = 0, 1, \ldots,$$

known as *Gauss-Seidel iteration method* (GS). Here $T_G = (D - L)^{-1}U$ is the *Gauss-Seidel iteration* matrix. Finally, from (2.5) we can write the equivalent system

$$(D - wL)x = (wU + (1 - w)D)x + wb$$

and for $w \in \mathbb{R}\backslash\{0\}$, it leads to the iterative method

$$x^{(k+1)} = (D - wL)^{-1} (wU + (1 - w) D) x^{(k)} + w (D - wL)^{-1} b, \quad k = 0, 1, \ldots,$$

known as the *Successive Overrelaxation method* (SOR). Here $w \in \mathbb{R}$ is called the *relaxation parameter* and $T_{SOR} = (D - wL)^{-1} (wU + (1 - w) D)$ is the iteration matrix. Notice that $w = 1$ gives the GS method.

To study the convergence of stationary iterative methods of the form (2.3), we need to analyze it, where $x^{(0)}$ is arbitrary.

The following two results give the basic and important instruments of such an analysis.

Lemma 2.3.1 *([2], pp.457). Let $T \in \mathcal{M}_n$. If $\rho(T) < 1$, then $(I - T)^{-1}$ exists and*

$$(I - T)^{-1} = I + T + T^2 + \cdots = \sum_{k=0}^{\infty} T^k.$$

Proof. Let λ be an eigenvale of T and x an eigenvector associated with the eigenvalue λ, i.e., $Tx = \lambda x$. Then $x - Tx = x - \lambda x$ or, equivalently, $(I - T)x = (1 - \lambda)x$ implies that $(1 - \lambda)$ is an eigenvalue of $(I - T)$. But since $|\lambda| \leq \rho(T) < 1$, $\lambda = 1$ is not an eigenvalue of T, and hence 0 cannot be an eigenvalue of $(I - T)$. Therefore, $(I - T)^{-1}$ exists. Let $S_m = I + T + T^2 + ... + T^m$. Then,

$$(I - T)S_m = (I + T + T^2 + \cdots + T^m) - (T + T^2 + \cdots + T^{m+1}) = I - T^{m+1},$$

and, since T is convergent, Theorem 2.2.20 implies that

$$\lim_{m\to\infty} (I - T)S_m = \lim_{m\to\infty} (I - T^{m+1}) = I.$$

Thus, $(I - T)^{-1} = \lim_{m\to\infty} S_m = I + T + T^2 + \cdots = \sum_{k=0}^{\infty} T^k$. □

Theorem 2.3.2 *([2], pp.457). For any $x^{(0)} \in \mathbb{R}^n$, the sequence $\{x^{(k)}\}_{k=0}^{\infty}$ defined by*

$$x^{(k)} = Tx^{(k-1)} + c \quad , \quad k = 1, 2, ... \tag{2.6}$$

converges to the unique solution of $x = Tx + c$ if and only if $\rho(T) < 1$.

Proof. First assume that $\rho(T) < 1$. Then,

$$\begin{aligned} x^{(k)} &= Tx^{(k-1)} + c \\ &= T(Tx^{(k-2)} + c) + c \\ &= T^2x^{(k-2)} + (T + I \quad)c \\ &\vdots \\ &= T^kx^{(0)} + (T^{k-1} + ... + T + I)c \end{aligned}$$

Since $\rho(T) < 1$, Theorem 2.2.20 implies that T is convergent, and

$$\lim_{k\to\infty} T^kx^{(0)} = 0.$$

Moreover, Lemma 2.3.1 implies that

$$\lim_{k\to\infty} x^{(k)} = \lim_{k\to\infty} T^kx^{(0)} + \left(\sum_{k=0}^{\infty} T^k\right)c = (I - T)^{-1}c.$$

Hence, the sequence $\{x^{(k)}\}_{k=0}^{\infty}$ converges to the vector $x = (I - T\ )^{-1}c$ and $x = Tx + c$.

Conversely, assume that the sequence $\{x^{(k)}\}_{k=0}^{\infty}$ defined by (2.6) converges to the unique solution for any $x^{(0)} \in \mathbb{R}$. It will be shown that for any $z \in \mathbb{R}^n$, we have $\lim_{k\to\infty} T^kz = 0$. By Theorem 2.2.20, this is equivalent to $\rho(T) < 1$. Let z be an arbitrary vector, and x be the unique solution to $x = Tx+c$. Define $x^{(0)} = x-z$, and for $k \geq 1$, $x^{(k)} = Tx^{(k-1)}+c$. Then, the sequence $\{x^{(k)}\}_{k=0}^{\infty}$ converges to x. Also,

$$x - x^{(k)} = (Tx + c) - (Tx^{(k-1)} + c) = T(x - x^{(k-1)})$$

and hence,

$$x - x^{(k)} = T(x - x^{(k-1)}) = T^2(x - x^{(k-2)}) = \cdots = T^k(x - x^{(0)}) = T^kz.$$

Thus, $\lim_{k\to\infty} T^kz = \lim_{k\to\infty} T^k(x - x^{(0)}) = \lim_{k\to\infty}(x - x^{(k)}) = 0$. Since $z \in \mathbb{R}^n$ is arbitrary, by Theorem 2.2.20, T is convergent and $\rho(T) < 1$. □

Corollary 2.3.3 *([2], pp.458).* *If $\|T\| < 1$ for any natural matrix norm and c is a given vector, then the sequence $\{x^{(k)}\}_{k=0}^{\infty}$ defined by $x^{(k)} = Tx^{(k-1)} + c$, $k = 1, 2, \ldots$ converges, for any $x^{(0)} \in \mathbb{R}^n$, to a vector $x \in \mathbb{R}^n$, with $x = Tx + c$, and the following error bounds hold:*

(a) $\left\|x - x^{(k)}\right\| \leq \|T\|^k \left\|x^{(0)} - x\right\|$.

(b) $\left\|x - x^{(k)}\right\| \leq \frac{\|T\|^k}{1-\|T\|} \left\|x^{(1)} - x^{(0)}\right\|$.

Since preconditioning for Jacobi and Gauss-Seidel iterations is our main concern, we may give the following results belonging to K.R. James [15].

Lemma 2.3.4 *([51], pp.121). An upper bound on the spectral radius of the Gauss-Seidel iteration matrix T given by*

$$\rho(T) \leq \max_i \frac{u_i}{1 - l_i}$$

for all $1 \leq i \leq n - 1$, *where* l_i *and* u_i *are the sums of modulus elements in row i of the triangular matrices L, U.*

Lemma 2.3.5 *([51], pp.121). If* $A = D - L - U$ *is a Z-matrix, then an upper bound on the spectral radius of the Gauss-Seidel iteration matrix T given by*

$$\rho(T) \leq \max_i \frac{u_i}{d_i - l_i} \tag{2.7}$$

for all i, where d_i, l_i *and* u_i *are the sums of modulus elements in row i of D, L and U, respectively.*

We have seen that the Jacobi and Gauss-Seidel iterative techniques can be written as

$$x^{(k)} = T_J x^{(k-1)} + c_j \text{ and } x^{(k)} = T_G x^{(k-1)} + c_g,$$

where $c_j = D^{-1}b$, $c_g = (D - L)^{-1}b$ and the iteration matrices T_J and T_G are given by

$$T_J = D^{-1}(L + U) \text{ and } T_G = (D - L)^{-1}U,$$

respectively. If $\rho(T_J) < 1$ or $\rho(T_G) < 1$, then the corresponding sequence $\{x^{(k)}\}_{k=0}^{\infty}$ will converge to the unique solution x^* of $Ax = b$. For example, the Jacobi scheme has

$$x^{(k)} = D^{-1}(L + U)x^{(k-1)} + D^{-1}b,$$

and, if $\{x^{(k)}\}_{k=0}^{\infty}$ converges to x^*, then

$$x^* = D^{-1}(L + U)x^* + D^{-1}b,$$

which implies that $Dx^* = (L + U)x^* + b$ or $(D - L - U)x^* = b$. Since $D - L - U = A$, the solution x^* satisfies $Ax = b$.

Definition 2.3.6 *([48], pp542). A real matrix $A \in \mathcal{M}_n$ is called strictly columnwise diagonally dominant (SCDD) if*

$$\left|a_{j,j}\right| > \sum_{i=1, i\neq j}^{n} \left|a_{i,j}\right| \text{ for } j = 1, 2, \ldots, n.$$

Similar definition can be given for strictly rowwise diagonally dominant matrices. Replacing the inequality sign $>$ with $\geq$, definitions for rowwise diagonally dominant and columnwise diagonally dominant (CDD) matrices are obtained.

Remark 2.3.7 *For the coefficient matrix $A \in \mathcal{M}_n$ of the linear system $Ax = b$, we consider splitting of the form $A = D - L - U$. Under the assumption that D is nonsingular, we have given Jacobi and Gauss-Seidel iteration matrices T_J and T_G by $T_J = D^{-1}(L + U)$ and $T_G = (D - L)^{-1} U$. Since D is nonsingular, without loss of generality, we assume that A has a splitting of the form $A = I - L - U$. Then Jacobi and Gauss-Seidel iterative methods turn into*

$$x^{(k+1)} = T_J x^{(k)} + c_j,\ k = 1, 2, \ldots, \tag{2.8}$$

where $T_J = L + U$, $c_j = b$ and

$$x^{(k+1)} = T_G x^{(k)} + c_g,\ k = 1, 2, \ldots, \tag{2.9}$$

where $T_G = (I - L)^{-1} U$, $c_g = (I - L)^{-1} b$. Unless specified otherwise, it will be assumed that nonsingular matrix A has a splitting of the form $A = I - L - U$.

Lemma 2.3.8 *([1]). Let $A = [a_{i,j}] = I - L - U \in \mathcal{M}_n$ be SCDD. Then A is nonsingular.*

Proof. Suppose that $A = [a_{i,j}]$ is SCDD but singular. Then there is a vector $u \neq \mathbf{0}$ such that

$$Au = \mathbf{0}.$$

The vector $u \neq \mathbf{0}$ has some entry $u_i > 0$ with the largest magnitude. Therefore

$$\begin{aligned}
\sum_{j=1}^{n} a_{i,j} u_j &= 0 \Longrightarrow \\
a_{i,i} u_i &= -\sum_{j=1}^{i-1} a_{i,j} u_j - \sum_{j=i+1}^{n} a_{i,j} u_j \Longrightarrow \\
u_i &= -\sum_{j=1}^{i-1} a_{i,j} u_j - \sum_{j=i+1}^{n} a_{i,j} u_j \Longrightarrow \\
1 &= -\sum_{j=1}^{i-1} a_{i,j} \frac{u_j}{u_i} - \sum_{j=i+1}^{n} a_{i,j} \frac{u_j}{u_i},
\end{aligned}$$

and hence

$$\begin{aligned}
1 &\leq \sum_{j=1}^{i-1} \left| a_{i,j} \frac{u_j}{u_i} \right| + \sum_{j=i+1}^{n} \left| a_{i,j} \frac{u_j}{u_i} \right| \\
&\leq \sum_{j=1}^{i-1} \left| a_{i,j} \right| + \sum_{j=i+1}^{n} \left| a_{i,j} \right|
\end{aligned}$$

since $\left| u_j / u_i \right| \leq 1$ for $j = 1, 2, \ldots, n$. But this is a contradiction. So, the proof is completed. □

Lemma 2.3.9 *([1]). Let $A = [a_{i,j}] = I - L - U \in \mathcal{M}_n$ be SCDD. For $|\lambda| \geq 1$, let $A_J(\lambda) = \lambda I - L - U$ and $A_G(\lambda) = \lambda (I - L) - U$. Then $A_J(\lambda)$ and $A_G(\lambda)$ are SCDD.*

Proof. For each $j = 1, 2, \ldots, n$ we have

$$\left| a_{j,j} \right| > \sum_{i=1, i \neq j}^{n} \left| a_{i,j} \right|,$$

or, since $a_{j,j} = 1$ for each $j = 1, 2 \ldots n$,

$$1 > \sum_{i=1, i \neq j}^{n} \left| a_{i,j} \right|.$$

Then,

$$\begin{aligned}
|\lambda| &> |\lambda| \sum_{i=1, i \neq j}^{n} \left| a_{i,j} \right| \\
&= |\lambda| \sum_{i=1}^{j-1} \left| a_{i,j} \right| + |\lambda| \sum_{i=j+1}^{n} \left| a_{i,j} \right| \\
&\geq |\lambda| \sum_{i=1}^{j-1} \left| a_{i,j} \right| + \sum_{i=j+1}^{n} \left| a_{i,j} \right|
\end{aligned}$$

which implies that $|\lambda| > \sum_{i=1}^{j-1} \left|a_{i,j}\right| + \sum_{i=j+1}^{n} \left|a_{i,j}\right|$ for each $j = 1, 2 \ldots, n$, that is $A_J(\lambda) = \lambda I - L - U$ is SCDD. Similarly

$$\begin{aligned} |\lambda| &> |\lambda| \sum_{i=1}^{j-1} \left|a_{i,j}\right| + \sum_{i=j+1}^{n} \left|a_{i,j}\right| \\ &= \sum_{i=1}^{j-1} \left|\lambda a_{i,j}\right| + \sum_{i=j+1}^{n} \left|a_{i,j}\right| \end{aligned}$$

implies that $|\lambda| > \sum_{i=1}^{j-1} \left|\lambda a_{i,j}\right| + \sum_{i=j+1}^{n} \left|a_{i,j}\right|$ for each $j = 1, 2 \ldots, n$, that is $A_G(\lambda) = \lambda (I - L) - U$ is SCDD. □

From Lemma 2.3.8 and Lemma 2.3.9 we can state the following.

Theorem 2.3.10 *([1]). Let $A = [a_{i,j}] = I - L - U \in \mathcal{M}_n$ be SCDD and let $A_J(\lambda) = \lambda I - L - U$ and $A_G(\lambda) = \lambda (I - L) - U$ where $|\lambda| \geq 1$. Then $A_J(\lambda)$ and $A_G(\lambda)$ are nonsingular.*

Now, having obtained the auxiliary results, we can state the main result.

Theorem 2.3.11 *([1]). Let $A = [a_{i,j}] = I - L - U \in \mathcal{M}_n$ be SCDD and consider Jacobi and Gauss-Seidel iteration matrices given by $T_J = L + U$ and $T_G = (I - L)^{-1} U$, respectively. Then $\rho(T_J) < 1$ and $\rho(T_G) < 1$.*

Proof. We have $p_{T_J}(\lambda) = \det(\lambda I - T_J)$. Let $A_J(\lambda) = \lambda I - L - U$. Then

$$\det(\lambda I - T_J) = \det(\lambda I - (L + U)) = \det(A_J(\lambda)).$$

So, for $p_{T_J}(\lambda) = \det(\lambda I - T_J) = 0$ to hold, we must have $\det(A_J(\lambda)) = 0$. But, since $A_J(\lambda)$ is nonsingular for $|\lambda| \geq 1$, it follows that all the eigenvalues of T_J have magnitude less than 1 and, hence, $\rho(T_J) < 1$. Similarly, for the Gauss-Seidel iteration matrix $T_G = (I - L)^{-1} U$ we have $p_{T_G}(\lambda) = \det(\lambda I - T_G)$. Let $A_G(\lambda) = \lambda (I - L) - U$. Then,

$$\begin{aligned} \det(\lambda I - T_G) &= \det\left(\lambda I - (I - L)^{-1} U\right) \\ &= \det\left(\lambda (I - L)^{-1} (I - L) - (I - L)^{-1} U\right) \\ &= \det\left((I - L)^{-1} (\lambda (I - L) - U)\right) \\ &= \det(I - L)^{-1} \det A_G(\lambda). \end{aligned}$$

So, for $p_{T_G}(\lambda) = \det(\lambda I - T_G) = 0$ to hold, we must have $\det A_G(\lambda) = 0$, since $\det(I-L)^{-1} \neq 0$. But, since $A_G(\lambda)$ is nonsingular for $|\lambda| \geq 1$, it follows that all the eigenvalues of T_G have magnitude less than 1 and, hence, $\rho(T_G) < 1$. □

Theorem 2.3.12 *([2], pp459)*. *Let* $A = \left[a_{i,j}\right] \in \mathcal{M}_n$. *If* $a_{i,j} \leq 0$, *for each* $i \neq j$ *and* $a_{i,i} > 0$, *for each* $i = 1, 2, \ldots, n$, *then one and only one of the following statements holds:*

(a) $0 \leq \rho(T_G) < \rho(T_J) < 1$,

(b) $1 < \rho(T_J) < \rho(T_G)$,

(c) $\rho(T_J) = \rho(T_G) = 0$,

(d) $\rho(T_J) = \rho(T_G) = 1$.

Definition 2.3.13 *([51]). Let* $A = \left[a_{i,j}\right]$, $B = \left[b_{i,j}\right] \in \mathcal{M}_n$. *Then* $A \geq B$ *if* $a_{i,j} \geq b_{i,j}$ *for all* $1 \leq i, j \leq n$.

Definition 2.3.14 *([51]).* $A = [a_{i,j}] \in \mathcal{M}_n$ *is an* $L-$*matrix if* $a_{i,i} > 0$, $i = 1, 2, ..., n$ *and* $a_{ij} \leq 0$ *for all* $i, j = 1, 2, \ldots, n; i \neq j$; *and it is an* $M-$*matrix if* A *has the form* $A = sI - B$, *where* $B \geq 0$ *and* $s \geqslant \rho(B)$.

Definition 2.3.15 *([8]).* $A = [a_{i,j}] \in \mathcal{M}_n$ *is a* $Z-$*matrix if* $a_{i,j} \leq 0$, $i, j = 1, 2, \ldots, n$; $i \neq j$ *and, a* $Z-$*matrix such that each column sum is equal to zero is called a* $Q-$*matrix.*

Definition 2.3.16 *([48], pp.360). For* $n \geq 2$, *a real matrix* $A \in \mathcal{M}_n$ *is reducible if there exists an nxn permutation matrix* P *such that*

$$PAP^T = \begin{bmatrix} A_{1,1} & A_{1,2} \\ \mathbf{0} & A_{2,2} \end{bmatrix}$$

where $A_{1,1}$ *is an* $r \times r$ *submatrix and* $A_{2,2}$ *is an* $(n-r) \times (n-r)$ *submatrix, where* $1 \leq r < n$. *If no such permutation matrix exists, then* A *is irreducible. If* A *is a* 1×1 *real matrix, then* A *is irreducible if its single entry is nonzero, and reducible otherwise.*

2.4 Preconditioning for Stationary Iterative Methods and Preconditioners

Consider the general form of the stationary iterative methods, such as Jacobi, Gauss-Seidel and SOR methods, given by

$$x^{(k+1)} = Tx^{(k)} + c, \quad k = 0, 1, \ldots,$$

where the aim is to find the unique solution x^* of the linear system $Ax = b$, $A = I - L - U$ and $x^{(0)}$ is the initial approximation. Consider the iteration matrix T. In the previous section we have seen that if $\rho(T) < 1$, then $\lim_{k \longrightarrow \infty} T^k x^{(0)} = \mathbf{0}$ for any $x^{(0)}$ and $\left\| x - x^{(k)} \right\| \leq \|T\|^k \left\| x - x^{(0)} \right\|$. It is obvious that smaller spectral radius $\rho(T)$ corresponds to faster convergence of the method, or less number of iterations for the same accuracy. So, the question is whether we can obtain an equivalent linear system for which iteration matrix has a smaller spectral radius for the stationary iterative method under consideration. For that reason, we consider the solution of the equivalent linear system

$$\widehat{P}Ax = \widehat{P}b, \tag{2.10}$$

where $\widehat{P}$ is nonsingular. The system (2.10) is the *preconditioned form* of $Ax = b$ and the nonsingular matrix $\widehat{P}$ is called the *preconditioning matrix* or *preconditioner*. So, the aim is to find a matrix $\widehat{P}$ such that $\rho\left(\widehat{T}\right) < \rho(T) < 1$.

For this purpose it has been offered too many different type of preconditioners for different matrix classes. To the author's knowledge the first such a preconditioner is the one offered by J.P. Milaszewicz in [35]: $P = (I + C)$, where

$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -a_{2,1} & 0 & \cdots & 0 \\ -a_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ -a_{n,1} & 0 & \cdots & 0 \end{bmatrix}.$$

The preconditioner in fact performs Gaussian elimination on the first column. It has been shown that it improves convergence of Jacobi and Gauss-Seidel methods for $Z-$

matrices. Then in [8] a preconditioner of the form $P = (I + S)$, where

$$S = \begin{bmatrix} 0 & -a_{1,2} & 0 & \cdots & 0 \\ 0 & 0 & -a_{2,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

has been offered by A.D. Gunawardena et al., which transforms the first upper codiagonal to zero. It has been shown that it improves Gauss-Seidel method for $Z-$matrices. Then In [41], the preconditioners $P = (I + R)$ and $P = (I + R + S)$, where

$$R = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ -a_{n,1} & -a_{n,2} & \dots & -a_{n,n-1} & 0 \end{bmatrix},$$

have been introduced. Parametric versions of the preconditioners $P = (I + C)$ and $P = (I + S)$ have been offered in [9], where $P = (I + C(\alpha))$ with

$$C(\alpha) = \begin{bmatrix} 0 & \dots & 0 & 0 \\ -\alpha_2 a_{2,1} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ -\alpha_{n-1} a_{n-1,1} & \dots & 0 & 0 \\ -\alpha_n a_{n,1} & \dots & 0 & 0 \end{bmatrix},$$

and in [19], where $P = I + S(\alpha)$ with

$$S(\alpha) = \begin{bmatrix} 0 & -\alpha_1 a_{1,2} & 0 & \dots & 0 \\ 0 & 0 & -\alpha_2 a_{2,3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\alpha_{n-1} a_{n-1,n} \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

In [22] the preconditioner $P = (I + S_{\max})$, where

$$(S_{max})_{i,j} = \begin{cases} -a_{i,k_{i,}} & j = k_i \\ 0 & \text{otherwise} \end{cases},$$

and $k_i = \min j \in \left\{ j : \max_{j(j>i)} \left| a_{i,j} \right| \right\}$, has been developed. Similar ones introduced in [34], where $P = (I + S_{\max}(\alpha))$ with

$$(S_{max}(\alpha))_{i,j} = \begin{cases} -\alpha_i a_{i,k_i,} & j = k_i \\ 0 & \text{otherwise} \end{cases},$$

and $k_i = \min j \in \left\{ j : \max_{j(j>i)} \left| a_{i,j} \right| \right\}$ and $\alpha_i \geq 0$, $i = 1, 2, \ldots, n-1$, and in [39] by Najafi, where $P = (I + S_m)$ with

$$(S_m)_{i,j} = \begin{cases} 0 & j \in Q_i \\ -a_{i,j} & \text{otherwise} \end{cases},$$

and $Q_i = \left\{ j : j < i \text{ and } \left| a_{i,j} \right| = \min_k \left| a_{i,k} \right| \right\}$. Another one which aims to eliminate one element per row by preconditioning is offered in [43] and it is given by $P = I + S_N$, where

$$S_N = \begin{bmatrix} 0 & & -a_{1,k_1} & & & \\ & 0 & & & & -a_{2,k_2} \\ & & 0 & & -a_{3,k_3} & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & \ddots \\ & & -a_{n,k_n} & & & 0 \end{bmatrix},$$

with $k_i \in \{1, 2, \ldots, i-1, i+1, \ldots, n\}$, $i = 1, 2, \ldots, n$. Wang et. al., [56] offered $P = (I + S_W(\alpha))$, where

$$S_W(\alpha) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ -\alpha_{k,1} a_{k,1} & -\alpha_{k,2} a_{k,2} & \cdots & -\alpha_{k,n} a_{k,n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

D.J. Evans [6] and Wang3 [54] introduced the preconditioners $P = (I + C_1)$ and

$P = (I + C_1(\alpha))$, respectively, where

$$C_1 = \begin{bmatrix} 0 & \dots & 0 & -a_{n,1} \\ 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix}$$

and

$$C_1(\alpha) = \begin{bmatrix} 0 & \dots & 0 & -\alpha a_{n,1} \\ 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix}.$$

For the preconditioners given above, the (coefficient) matrices are, in general, $Z-$matrices, $L-$matrices, $M-$matrices and nonnegative matrices.

Linear combinations of some of the preconditioners given above have also been offered as preconditioners. Some of them are; $P_S = (I + S)$, $P_R = (I + R)$, $P_1 = (I + C_1)$, $P_{S1} = (I + S + C_1)$, $P_{RU} = (I + R + U)$ and $P_{SR} = (I + S + R)$.

A structurally quite different preconditioner from the ones given above is offered by M.Usui et.al. in [51], initially for $Z-$matrices. It is given by $P = (I + U)$, where $-U$ is the strictly upper triangular part of $A = I - L - U$. Then H.Kotakemori et. al., offered the preconditioner $P = (I + \beta U)$, where $\beta \in \mathbb{R}\backslash\{0\}$. It has also been studied in [44].

CHAPTER 3

NEW PRECONDITIONERS

We consider the solution of the linear systems of the form

$$Ax = b,$$

by stationary iterative methods, where the coefficient matrix $A = \left[a_{i,j}\right] \in \mathcal{M}_n$ is *strictly columnwise diagonally dominant* (SCDD). Without loss of generality, assume that $a_{i,i} = 1$ for $i = 1, 2, \ldots, n$ and consider the most general $n \times n$ SCDD matrix A with *unit diagonal entries*:

$$A = \begin{bmatrix} 1 & a_{1,2} & \cdots & a_{1,m} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & 1 & \cdots & a_{2,m} & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & 1 & \cdots & a_{m,n-1} & a_{m,n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,m} & \cdots & 1 & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} & \cdots & a_{n,n-1} & 1 \end{bmatrix}, \qquad (3.1)$$

We offer new preconditioners P_P of the form $P_P = I + \sum_{i=1}^{l} P_{n_i}$, where $n_i \in \{1, 2, \ldots, n\}$ with $n_i \neq n_j$ if $i \neq j$, $l \in \{1, 2, \ldots, n\}$ and for $n_i = m \in \{1, 2, \ldots, n\}$,

$$P_m = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ w_m & \cdots & w_m & 0 & w_m & \cdots & w_m \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where the real parameter w_m, that will be determined, appears in the mth row only. If $l = 1$, then the preconditioning is performed on a single row, that is on row n_1. If, on the other hand $1 < l < n$, then the preconditioning is performed on the rows $n_1, n_2, \ldots, n_{l-1}$ and n_l. In P_P, subscript P is used to denote *partial preconditioning*, that is preconditioning on a limited number of rows. If $l = n$, then all the rows of A will be preconditioned and in this case instead of P_P, the notation P_C will be used to denote *complete preconditioning*. Notice that, for $l = 1$ and $n_1 = m$, the preconditioned matrix $\widetilde{A}_m = (I + P_m)A = \left[\widetilde{a}_{i,j}\right]$ will be of the form

$$\widetilde{A}_m = \begin{bmatrix} 1 & a_{1,2} & \cdots & a_{1,m} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & 1 & \cdots & a_{2,m} & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \widetilde{a}_{m,1} & \widetilde{a}_{m,2} & \cdots & \widetilde{a}_{m,m} & \cdots & \widetilde{a}_{m,n-1} & \widetilde{a}_{m,n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,m} & \cdots & 1 & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} & \cdots & a_{n,n-1} & 1 \end{bmatrix},$$

where the mth row entries are given by

$$\widetilde{a}_{m,j} = a_{m,j} + w_m \left(\sum_{i=1}^{n} a_{i,j} - a_{m,j} \right), \quad j = 1, 2, \ldots, n.$$

For $j = m$ we have $\widetilde{a}_{m,m} = 1 + w_m \left(\sum_{i=1}^{n} a_{i,m} - 1 \right)$. So, $\widetilde{A}_m = (I + P_m)A$ and A differ only in the mth rows and hence, $\widetilde{A}_P = (I + P_P)A$ and A differ only in the rows $n_1, n_2, \ldots, n_l$, and as a result $\widetilde{A}_C = (I + P_C)A$ and A differ in the rows $1, 2, \ldots, n$. For that reason, we perform comparisons between the mth rows of the coefficient matrices of the given linear system and the preconditioned linear system, and then generalize the result.

For the positiveness of the diagonal entries of $\widetilde{A}_m = (I + P_m)A$, one needs the condition $1 + w_m \left(\sum_{i=1}^{n} a_{i,m} - 1 \right) > 0$ or

$$w_m \left(\sum_{i=1}^{n} a_{i,m} - 1 \right) > -1. \tag{3.2}$$

Then, we multiply $\widetilde{A}_m = (I + P_m)A$ by $\widehat{D}_m$ to make the mth diagonal entry 1, where $\widehat{D}_m$ is the diagonal matrix given by

$$\widehat{D}_m = \operatorname{diag}\left(1, ..., 1, \left(1 + w_m \left(\sum_{i=1}^{n} a_{i,m} - 1 \right) \right)^{-1}, 1, \ldots, 1 \right),$$

where $(1 + w_m(\sum_{i=1}^{n} a_{i,m} - 1))^{-1}$ is the mth diagonal entry. So, the entries of the mth row of the matrix $\widehat{A}_m = \widehat{D}_m (I + P_m) A$ are given by

$$\widehat{a}_{m,j} = \frac{w_m \left(\sum\limits_{i=1}^{n} a_{i,j} - a_{m,j} \right) + a_{m,j}}{1 + w_m \left(\sum\limits_{i=1}^{n} a_{i,m} - 1 \right)}, \quad \text{for } j = 1, 2, \ldots, n.$$

Recalling the structure of the iteration matrices of Jacobi and Gauss-Seidel iterative methods defined by (2.8) and (2.9), respectively, we consider the following well-known results which will be used in developing our new preconditioners.

Theorem 3.0.1 *([10], pp491). Let $T_1, T_2 \in \mathcal{M}_n$. If $|T_1| \leqslant T_2$, then $\rho(T_1) \leqslant \rho(|T_1|) \leqslant \rho(T_2)$.*

Corollary 3.0.2 *([10], pp491). Let $T_1, T_2 \in \mathcal{M}_n$. If $0 \leqslant T_1 \leqslant T_2$, then $\rho(T_1) \leqslant \rho(T_2)$.*

Let T and $\widehat{T}$ be the iteration matrices associated with the linear system $Ax = b$ and the equivalent preconditioned linear system $\widehat{A}_m x = \widehat{b}_m$, respectively, where $\widehat{A}_m = \widehat{D}_m(I + P_m)A$ and $\widehat{b}_m = \widehat{D}_m(I + P_m)b$. For some classes of matrices, our aim is to determine the parameter $w_m \in \mathbb{R}$ in such a way that $\rho\left(\widehat{T}\right) \leq \rho(T) < 1$ and hence to speed up the convergence of the stationary iterative methods under consideration.

Throughout the thesis we mainly consider two classes of coefficient matrices; SCDD $L-$ matrices and SCDD positive matrices .

3.1 Preconditioners for Linear Systems with SCDD $L-$Matrices

Let A be a SCDD $L-$matrix with unit diagonal entries and negative off-diagonal entries and let the preconditioned matrix $\overline{A}_m$ be defined by $\overline{A}_m = [\overline{a}_{i,j}] = \overline{D}_m(I + P_m)A$, where $\overline{D}_m = \text{diag}\left(1, \ldots, 1, (1 + w_m (\sum_{i=1}^{n} a_{i,m} - 1))^{-1}, 1, \ldots, 1\right)$. Then we have the following.

Lemma 3.1.1 *If* $w_m = w_{m,k} = \min_{1\le j\le n,\ j\ne m} \left\{w_{m,j}\right\}$, *where*

$$w_{m,j} = -a_{m,j}/\left(\sum\nolimits_{i=1}^{n} a_{i,j} - a_{m,j}\right), \tag{3.3}$$

then $a_{m,j} \le \overline{a}_{m,j} \le 0$ *for* $j = 1, 2, \ldots, n$, $j \ne m$ *and* $|\overline{A}_m| \le |A|$.

Proof. First of all, let us show that $w_{m,k}$ satisfies the inequality (3.2) for $w_m = w_{m,k}$. Since A is a SCDD $L-$matrix, we have $a_{m,k} \sum_{i=1}^{n} a_{i,m} < 0 < \sum_{i=1}^{n} a_{i,k}$. So,

$$a_{m,k}\left(\sum_{i=1}^{n} a_{i,m} - 1\right) < \sum_{i=1}^{n} a_{i,k} - a_{m,k}.$$

Then, dividing all sides by $-\left(\sum_{i=1}^{n} a_{i,k} - a_{m,k}\right)\left(\sum_{i=1}^{n} a_{i,m} - 1\right) > 0$ we get

$$w_{m,k} = -\frac{a_{m,k}}{\sum_{i=1}^{n} a_{i,k} - a_{m,k}} < -\frac{1}{\sum_{i=1}^{n} a_{i,m} - 1},$$

which is the requirement.

We have

$$\overline{a}_{m,j} = \frac{w_m\left(\sum\limits_{i=1}^{n} a_{i,j} - a_{m,j}\right) + a_{m,j}}{1 + w_m\left(\sum\limits_{i=1}^{n} a_{i,m} - 1\right)} \quad \text{for } j = 1, 2, \ldots, n.$$

Then taking $w_m = w_{m,k} = -a_{m,k}/\left(\sum_{i=1}^{n} a_{i,k} - a_{m,k}\right)$, we obtain

$$\overline{a}_{m,j} = \frac{-a_{m,k}\left(\sum\limits_{i=1}^{n} a_{i,j} - a_{m,j}\right) + a_{m,j}\left(\sum\limits_{i=1}^{n} a_{i,k} - a_{m,k}\right)}{-a_{m,k}\left(\sum\limits_{i=1}^{n} a_{i,m} - 1\right) + \sum\limits_{i=1}^{n} a_{i,k} - a_{m,k}}$$

or simplifying,

$$\overline{a}_{m,j} = \frac{a_{m,j}\sum\limits_{i=1}^{n} a_{i,k} - a_{m,k}\sum\limits_{i=1}^{n} a_{i,j}}{\sum\limits_{i=1}^{n} a_{i,k} - a_{m,k}\sum\limits_{i=1}^{n} a_{i,m}} \quad \text{for } j = 1, 2, \ldots, n.$$

Notice that for $j = m$ one has

$$\overline{a}_{m,m} = \frac{\sum\limits_{i=1}^{n} a_{i,k} - a_{m,k}\sum\limits_{i=1}^{n} a_{i,m}}{\sum\limits_{i=1}^{n} a_{i,k} - a_{m,k}\sum\limits_{i=1}^{n} a_{i,m}} = 1 > 0. \tag{3.4}$$

Since A is a SCDD $L-$matrix, we have $\sum_{i=1}^{n} a_{i,m} > 0$ which leads to

$$-a_{m,k}\left(\sum_{i=1}^{n} a_{i,m} - 1\right) > a_{m,k}.$$

Then adding $(\sum_{i=1}^{n} a_{i,k} - a_{m,k})$ to both sides of the inequality one obtains

$$\left(\sum_{i=1}^{n} a_{i,k} - a_{m,k}\right) - a_{m,k}\left(\sum_{i=1}^{n} a_{i,m} - 1\right) > \sum_{i=1}^{n} a_{i,k} - a_{m,k} + a_{m,k}$$

or

$$\sum_{i=1}^{n} a_{i,k} - a_{m,k} \sum_{i=1}^{n} a_{i,m} > 0. \tag{3.5}$$

For $j = 1, 2, \ldots, n,\ j \neq m$ we have

$$-\frac{a_{m,k}}{\sum\limits_{i=1}^{n} a_{i,k} - a_{m,k}} \leq -\frac{a_{m,j}}{\sum\limits_{i=1}^{n} a_{i,j} - a_{m,j}}. \tag{3.6}$$

Since $\sum\limits_{i=1}^{n} a_{i,k} - a_{m,k} > 0$ and $\sum\limits_{i=1}^{n} a_{i,j} - a_{m,j} > 0$, from (3.6) it follows that

$$-a_{m,k}\left(\sum_{i=1}^{n} a_{i,j} - a_{m,j}\right) \leq -a_{m,j}\left(\sum_{i=1}^{n} a_{i,k} - a_{m,k}\right)$$

or simplifying,

$$a_{m,j} \sum_{i=1}^{n} a_{i,k} - a_{m,k} \sum_{i=1}^{n} a_{i,j} \leq 0 \ \text{ for } j = 1, 2, \ldots, n, j \neq m. \tag{3.7}$$

Hence (3.4), (3.5) and (3.7) altogether show that $\overline{A}_m$ is an $L-$matrix.

Due to the strict diagonal dominancy of the $L-$matrix A, for $j = 1, 2, \ldots, n,\ j \neq m$, one has

$$\sum_{i=1}^{n} a_{i,j} > 0 > a_{m,j} \sum_{i=1}^{n} a_{i,m}$$

and hence, for $j = 1, 2, \ldots, m-1, m+1, \ldots, n$ one gets

$$a_{m,k} \sum_{i=1}^{n} a_{i,j} < a_{m,k}.a_{m,j} \sum_{i=1}^{n} a_{i,m}.$$

So, adding $-a_{m,j} \sum\limits_{i=1}^{n} a_{i,k}$ to both sides of the inequality, we obtain

$$-a_{m,j} \sum_{i=1}^{n} a_{i,k} + a_{m,k} \sum_{i=1}^{n} a_{i,j} < -a_{m,j} \sum_{i=1}^{n} a_{i,k} + a_{m,k}.a_{m,j} \sum_{i=1}^{n} a_{i,m},$$

or

$$-\left(a_{m,j} \sum_{i=1}^{n} a_{i,k} - a_{m,k} \sum_{i=1}^{n} a_{i,j}\right) < -a_{m,j}\left(\sum_{i=1}^{n} a_{i,k} - a_{m,k} \sum_{i=1}^{n} a_{i,m}\right).$$

Then, dividing all sides by $(\sum_{i=1}^{n} a_{i,k} - a_{m,k} \sum_{i=1}^{n} a_{i,m}) > 0$ we get

$$-\overline{a}_{mj} = -\frac{\left(a_{m,j} \sum_{i=1}^{n} a_{i,k} - a_{m,k} \sum_{i=1}^{n} a_{i,j}\right)}{\left(\sum_{i=1}^{n} a_{i,k} - a_{m,k} \sum_{i=1}^{n} a_{i,m}\right)} < -a_{m,j} \text{ for } j = 1, 2, \ldots, n,\ j \neq m,$$

which is the required inequality, that is $|\overline{A}_m| \leq |A|$. □

Lemma 3.1.1 holds for $m = n_1, n_2, \ldots, n_l$, where $l \in \{1, 2, \ldots, n\}$. Therefore for SCDD $L-$matrices having at least one row of negative off-diagonal entries we can state the following.

Theorem 3.1.2 *Let $A = [a_{i,j}]$ be a SCDD $L-$matrix of the form (3.1) having at least one row of negative off-diagonal entries and let $\overline{A}_C = \left[\overline{a}_{i,j}\right] = \overline{D}_C (I + P_C) A$, where P_C is given by*

$$P_C = \begin{bmatrix} 0 & w_1 & \cdots & \cdots & w_1 \\ w_2 & 0 & \cdots & \cdots & w_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{n-1} & w_{n-1} & \cdots & 0 & w_{n-1} \\ w_n & w_n & \cdots & w_n & 0 \end{bmatrix}.$$

and $\overline{D}_C = diag\Big((1 + w_1 (\sum_{i=1}^{n} a_{i,1} - 1))^{-1}, \ldots, (1 + w_n (\sum_{i=1}^{n} a_{i,n} - 1))^{-1}\Big)$. Moreover, let $w_l = w_{l,k_l} = \min_{1 \leq j \leq n,\ j \neq l} \left\{w_{l,j}\right\}$ for $l = 1, 2, \ldots, n$, where

$$w_{l,j} = -a_{l,j} / \left(\sum\nolimits_{i=1}^{n} a_{i,j} - a_{l,j}\right).$$

Then $a_{i,j} \leq \overline{a}_{i,j} \leq 0$ for $i, j = 1, 2, \ldots, n, i \neq j$, and $|\overline{A}_C| \leq |A|$.

Let A be a SCDD $L-$matrix with unit diagonal entries and negative off-diagonal entries, $\overline{\overline{A}}_m = [\overline{\overline{a}}_{i,j}] = \overline{\overline{D}}_m(I + P_m)A$, where $\overline{\overline{D}}_m$ is the diagonal matrix given by $\overline{\overline{D}}_m = \text{diag}\Big(1, \ldots, 1, (1 + w_m (\sum_{i=1}^{n} a_{i,m} - 1))^{-1}, 1, \ldots, 1\Big)$. Then we have the following.

Lemma 3.1.3 *If $w_m = w_{m,k} = \min_{1 \leq j \leq n,\ j \neq m} \left\{w_{m,j}\right\}$, where*

$$w_{m,j} = -2a_{m,j} / \left(\sum\nolimits_{i=1}^{n} a_{i,j} + a_{m,j} \sum\nolimits_{i=1}^{n} a_{i,m} - 2a_{m,j}\right), \tag{3.8}$$

then $a_{m,j} \leq \overline{\overline{a}}_{m,j} \leq -a_{m,j}$ for $j = 1, 2, \ldots, n, j \neq m$, and $|\overline{\overline{A}}_m| \leq |A|$.

Proof. First, we need to show that $w_{m,k}$ defined by (3.8) satisfies the inequality (3.2) for $w_m = w_{m,k}$. One has $-a_{m,k}\sum_{i=1}^n a_{i,m} > 0 > -\sum_{i=1}^n a_{i,k}$ which leads to

$$-2a_{m,k}\left(\sum_{i=1}^n a_{i,m}-1\right) > -\left(\sum_{i=1}^n a_{i,k}+a_{m,k}\sum_{i=1}^n a_{i,m}-2a_{m,k}\right).$$

Dividing both sides by $(\sum_{i=1}^n a_{i,k}+a_{m,k}\sum_{i=1}^n a_{i,m}-2a_{m,k}) > 0$, we get

$$w_{m,k}\left(\sum_{i=1}^n a_{i,m}-1\right) = -\frac{2a_{m,k}\left(\sum_{i=1}^n a_{i,m}-1\right)}{\sum_{i=1}^n a_{i,k}+a_{m,k}\sum_{i=1}^n a_{i,m}-2a_{m,k}} > -1,$$

that is the desired inequality.

We have

$$\overline{\overline{a}}_{m,j} = \frac{w_m\left(\sum\limits_{i=1}^n a_{i,j}-a_{m,j}\right)+a_{m,j}}{1+w_m\left(\sum\limits_{i=1}^n a_{i,m}-1\right)}, \quad \text{for } j=1,2,\ldots,n.$$

Then taking $w_m = w_{m,k} = -2a_{m,k}/\left(a_{m,k}\sum_{i=1}^n a_{i,m}+\sum_{i=1}^n a_{i,k}-2a_{m,k}\right)$, we obtain

$$\overline{\overline{a}}_{m,j} = \frac{-2a_{m,k}\left(\sum\limits_{i=1}^n a_{i,j}-a_{m,j}\right)+a_{m,j}\left(\left(\left(\sum\limits_{i=1}^n a_{i,k}\right)-a_{m,k}\right)+a_{m,k}\left(\sum\limits_{i=1}^n a_{i,m}-1\right)\right)}{\left(\left(\left(\sum\limits_{i=1}^n a_{i,k}\right)-a_{m,k}\right)+a_{m,k}\left(\sum\limits_{i=1}^n a_{i,m}-1\right)\right)-2a_{m,k}\left(\sum\limits_{i=1}^n a_{i,m}-1\right)}$$

or simplifying,

$$\overline{\overline{a}}_{m,j} = \frac{-2a_{m,k}\sum\limits_{i=1}^n a_{i,j}+a_{m,j}\left(\sum\limits_{i=1}^n a_{i,k}+a_{m,k}\sum\limits_{i=1}^n a_{i,m}\right)}{\sum\limits_{i=1}^n a_{i,k}-a_{m,k}\sum\limits_{i=1}^n a_{i,m}} \qquad \text{for} \qquad j=1,2,\ldots,n.$$

Notice that for $j=m$ one gets

$$\overline{\overline{a}}_{m,m} = \frac{-2a_{m,k}\sum\limits_{i=1}^n a_{i,m}+\left(\sum\limits_{i=1}^n a_{i,k}+a_{m,k}\sum\limits_{i=1}^n a_{i,m}\right)}{\sum\limits_{i=1}^n a_{i,k}-a_{m,k}\sum\limits_{i=1}^n a_{i,m}}$$

or

$$\overline{\overline{a}}_{m,m} = \frac{\sum\limits_{i=1}^n a_{i,k}-a_{m,k}\sum\limits_{i=1}^n a_{i,m}}{\sum\limits_{i=1}^n a_{i,k}-a_{m,k}\sum\limits_{i=1}^n a_{i,m}} = 1.$$

On the other hand, for $j=1,2,\ldots,n$, $j\neq m$, we have

$$a_{m,j}\sum_{i=1}^n a_{i,m} < 0 < \sum_{i=1}^n a_{i,j}$$

which lead to

$$-2a_{m,k}a_{m,j}\sum_{i=1}^{n}a_{i,m} < -2a_{m,k}\sum_{i=1}^{n}a_{i,j}$$

or

$$-a_{m,j}a_{m,k}\sum_{i=1}^{n}a_{i,m} - a_{m,j}a_{m,k}\sum_{i=1}^{n}a_{i,m} < -2a_{m,k}\sum_{i=1}^{n}a_{i,j}.$$

So, adding $a_{m,j}\sum_{i=1}^{n}a_{i,k}$ to both sides of the last inequality we obtain

$$a_{m,j}\sum_{i=1}^{n}a_{i,k} - a_{m,j}a_{m,k}\sum_{i=1}^{n}a_{i,m} - a_{m,j}a_{m,k}\sum_{i=1}^{n}a_{i,m} < -2a_{m,k}\sum_{i=1}^{n}a_{i,j} + a_{m,j}\sum_{i=1}^{n}a_{i,k},$$

or

$$a_{m,j}\left(\sum_{i=1}^{n}a_{i,k} - a_{m,k}\sum_{i=1}^{n}a_{i,m}\right) < -2a_{m,k}\sum_{i=1}^{n}a_{i,j} + a_{m,j}\left(\sum_{i=1}^{n}a_{i,k} + a_{m,k}\sum_{i=1}^{n}a_{i,m}\right).$$

Finally, dividing all sides by $(\sum_{i=1}^{n}a_{i,k} - a_{m,k}\sum_{i=1}^{n}a_{i,m}) > 0$ we get

$$a_{m,j} < \frac{-2a_{m,k}\sum_{i=1}^{n}a_{i,j} + a_{m,j}\left(\sum_{i=1}^{n}a_{i,k} + a_{m,k}\sum_{i=1}^{n}a_{i,m}\right)}{\left(\sum_{i=1}^{n}a_{i,k} - a_{m,k}\sum_{i=1}^{n}a_{i,m}\right)} = \overline{\overline{a}}_{m,j}$$

or

$$\overline{\overline{a}}_{m,j} > a_{m,j} \text{ for } j = 1, 2, \ldots, n,\ j \neq m. \tag{3.9}$$

Now, we shall see $\overline{\overline{a}}_{m,j} \leq -a_{m,j}$ for $j = 1, 2, \ldots, n$, $j \neq m$, and complete the proof.

From $w_{m,k} = \min_{1\leq j\leq n, j\neq m}\left\{-2a_{m,j}/\left(a_{m,j}\sum_{i=1}^{n}a_{i,m} + \sum_{i=1}^{n}a_{i,j} - 2a_{m,j}\right)\right\}$ for $j = 1, 2, \ldots, n$, $j \neq m$, we obtain

$$\frac{-2a_{m,k}}{\sum_{i=1}^{n}a_{i,k} + a_{m,k}\sum_{i=1}^{n}a_{i,m} - 2a_{m,k}} \leq \frac{-2a_{m,j}}{\sum_{i=1}^{n}a_{i,j} + a_{m,j}\sum_{i=1}^{n}a_{i,m} - 2a_{m,j}}.$$

Since $\left(\sum_{i=1}^{n}a_{i,j}\right) > 0$, $a_{m,j}((\sum_{i=1}^{n}a_{i,m}) - 1) \geq 0$ and $-a_{m,j} \geq 0$, it follows that

$$\left(\sum_{i=1}^{n}a_{i,j} + a_{m,j}\left(\sum_{i=1}^{n}a_{i,m} - 1\right) - a_{m,j}\right) > 0.$$

Then, from (3.1) we get

$$-2a_{m,k}\left(\sum_{i=1}^{n}a_{i,j} + a_{m,j}\sum_{i=1}^{n}a_{i,m} - 2a_{m,j}\right) \leq -2a_{m,j}\left(\sum_{i=1}^{n}a_{i,k} + a_{m,k}\sum_{i=1}^{n}a_{i,m} - 2a_{m,k}\right)$$

or

$$a_{m,k}\sum_{i=1}^{n}a_{i,j}+a_{m,k}a_{m,j}\sum_{i=1}^{n}a_{i,m}-2a_{m,k}a_{m,j}\geq a_{m,j}\sum_{i=1}^{n}a_{i,k}+a_{m,j}a_{m,k}\sum_{i=1}^{n}a_{i,m}-2a_{m,j}a_{m,k}$$

and simplifying

$$a_{m,k}\sum_{i=1}^{n}a_{i,j}-a_{m,j}\sum_{i=1}^{n}a_{i,k}\geq 0 \text{ for } j=1,2,\ldots,n,\ j\neq m. \tag{3.10}$$

Then, the inequality (3.10) leads to

$$-2a_{m,k}\sum_{i=1}^{n}a_{i,j}+a_{m,j}\sum_{i=1}^{n}a_{i,k}\leq -a_{m,j}\sum_{i=1}^{n}a_{i,k}$$

or, adding $a_{m,k}a_{m,j}\sum_{i=1}^{n}a_{i,m}$ to both sides of the last inequality one obtains

$$-2a_{m,k}\sum_{i=1}^{n}a_{i,j}+a_{m,k}a_{m,j}\sum_{i=1}^{n}a_{i,m}+a_{m,j}\sum_{i=1}^{n}a_{i,k}\leq a_{m,k}a_{m,j}\sum_{i=1}^{n}a_{i,m}-a_{m,j}\sum_{i=1}^{n}a_{i,k}$$

or

$$-2a_{m,k}\sum_{i=1}^{n}a_{i,j}+a_{m,j}\left(a_{m,k}\sum_{i=1}^{n}a_{i,m}+\sum_{i=1}^{n}a_{i,k}\right)\leq -a_{m,j}\left(\sum_{i=1}^{n}a_{i,k}-a_{m,k}\sum_{i=1}^{n}a_{i,m}\right).$$

Hence, dividing all sides by $\sum_{i=1}^{n}a_{i,k}-a_{m,k}\sum_{i=1}^{n}a_{i,m}>0$ we get

$$\overline{\overline{a}}_{m,j}=\frac{-2a_{m,k}\sum\limits_{i=1}^{n}a_{i,j}+a_{m,j}\left(a_{m,k}\sum_{i=1}^{n}a_{i,m}+\sum\limits_{i=1}^{n}a_{i,k}\right)}{\sum\limits_{i=1}^{n}a_{i,k}-a_{m,k}\sum_{i=1}^{n}a_{i,m}}\leq -a_{m,j}. \tag{3.11}$$

Finally, (3.9) and (3.11) yield $a_{m,j}\leq\overline{\overline{a}}_{m,j}\leq -a_{m,j}$ for $j=1,2,\ldots,n,\ j\neq m$, and hence, $|\overline{\overline{A}}_m|\leq|A|$. □

Since Lemma 3.1.3 holds for $m=n_1,n_2,\ldots,n_l$, where $l\in\{1,2,\ldots,n\}$, we can state the following.

Theorem 3.1.4 *Let $A=\left[a_{i,j}\right]$ be a SCDD L−matrix of the form (3.1) having at least one row of negative off-diagonal entries and let $\overline{\overline{A}}_C=\left[\overline{\overline{a}}_{i,j}\right]=\overline{\overline{D}}_C\left(I+P_C\right)A$, where P_C is given by*

$$P_C=\begin{bmatrix}0 & w_1 & \cdots & \cdots & w_1\\ w_2 & 0 & \cdots & \cdots & w_2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ w_{n-1} & w_{n-1} & \cdots & 0 & w_{n-1}\\ w_n & w_n & \cdots & w_n & 0\end{bmatrix}$$

and $\overline{\overline{D}}_C$ =*diag*$\Big((1 + w_1 (\sum_{i=1}^n a_{i,1} - 1))^{-1}, \ldots, (1 + w_n (\sum_{i=1}^n a_{i,n} - 1))^{-1}\Big)$. *Moreover, let* $w_l = w_{l,k_l} = \min_{1 \le j \le n,\ j \ne l} \{w_{l,j}\}$ *for* $l = 1, 2, \ldots, n$, *where*

$$w_{l,j} = -2a_{l,j} / \left(\sum_{i=1}^n a_{i,j} + a_{l,j} \sum_{i=1}^n a_{i,l} - a_{l,j} \right).$$

Then $a_{i,j} \le \overline{\overline{a}}_{i,j} < -a_{i,j}$ *for* $i, j = 1, 2, \ldots, n$, $i \ne j$, *and* $|\overline{\overline{A}}_C| \le |A|$.

3.2 Preconditioners for Linear Systems with SCDD Positive Mat rices

Let $A > \mathbf{0}$ be a SCDD matrix with unit diagonal entries, $\overline{A}_m = [\overline{a}_{i,j}] = \overline{D}_m (I + P_m) A$ and $\overline{D}_m$ =diag$\Big(1, \ldots, 1, (1 + w_m (\sum_{i=1}^n a_{i,m} - 1))^{-1}, 1, \ldots, 1\Big)$. Then, we have the following.

Lemma 3.2.1 *If* $w_m = w_{m,k} = \max_{1 \le j \le n,\ j \ne m} \{w_{m,j}\}$, *where*

$$w_{m,j} = -a_{m,j} / \left(\sum_{i=1}^n a_{i,j} - a_{m,j} \right)$$

then $0 \le \overline{a}_{m,j} \le a_{m,j}$ *for* $j = 1, 2, \ldots, n$, $j \ne m$ *and* $\mathbf{0} \le \overline{A}_m \le A$.

Proof. One has

$$\overline{a}_{m,j} = \frac{w_m \left(\sum_{i=1}^n a_{i,j} - a_{m,j} \right) + a_{m,j}}{1 + w_m \left(\sum_{i=1}^n a_{i,m} - 1 \right)}, \quad \text{for } j = 1, 2, \ldots, n.$$

Then, taking $w_m = w_{m,k} = -a_{m,k} / (\sum_{i=1}^n a_{i,k} - a_{m,k})$ for $j = 1, 2, \ldots, n$, we obtain

$$\overline{a}_{m,j} = \frac{-a_{m,k} \left(\sum_{i=1}^n a_{i,j} - a_{m,j} \right) + a_{m,j} \left(\sum_{i=1}^n a_{i,k} - a_{m,k} \right)}{-a_{m,k} \left(\sum_{i=1}^n a_{i,m} - 1 \right) + \sum_{i=1}^n a_{i,k} - a_{m,k}}$$

or simplifying,

$$\overline{a}_{m,j} = \frac{a_{m,j} \sum_{i=1}^n a_{i,k} - a_{m,k} \sum_{i=1}^n a_{i,j}}{\sum_{i=1}^n a_{i,k} - a_{m,k} \sum_{i=1}^n a_{i,m}}.$$

Notice that for $j = m$ one gets

$$\overline{a}_{m,m} = \frac{\sum_{i=1}^{n} a_{i,k} - a_{m,k} \sum_{i=1}^{n} a_{i,m}}{\sum_{i=1}^{n} a_{i,k} - a_{m,k} \sum_{i=1}^{n} a_{i,m}} = 1.$$

First of all, we shall prove that $\overline{A}_m$ is a SCDD nonnegative matrix. Since A is SCDD positive matrix, for $j = 1, 2, \ldots, n$, $j \neq m$, it follows that $0 < \sum_{i=1}^{n} a_{i,m} - 1 < 1$ and $1 < \sum_{i=1}^{n} a_{i,j} - a_{m,j}$. So, $0 < \sum_{i=1}^{n} a_{i,m} - 1 < \sum_{i=1}^{n} a_{i,j} - a_{m,j}$ or $0 < a_{m,j} \left(\sum_{i=1}^{n} a_{i,j} - a_{m,j}\right)^{-1} < \left(\sum_{i=1}^{n} a_{i,j} - a_{m,j}\right)^{-1} < (\sum_{i=1}^{n} a_{i,m} - 1)^{-1}$ or

$$-\frac{1}{\sum_{i=1}^{n} a_{i,m} - 1} < -\frac{a_{m,j}}{\sum_{i=1}^{n} a_{i,j} - a_{m,j}} < 0 \text{ for } j = 1, 2, \ldots, n, j \neq m. \tag{3.12}$$

Notice that, if we take $j = k$ in the last inequality and multiply it by $(\sum_{i=1}^{n} a_{i,m} - 1) > 0$ we get the inequality (3.2). So, multiplying both sides of the inequality (3.12) by $(\sum_{i=1}^{n} a_{i,m} - 1)\left(\sum_{i=1}^{n} a_{i,j} - a_{m,j}\right) > 0$, one gets

$$\sum_{i=1}^{n} a_{i,j} - a_{m,j} \sum_{i=1}^{n} a_{i,m} > 0$$

for $j = 1, 2, \ldots, n$, $j \neq m$. Then, $j = k$ leads to

$$\sum_{i=1}^{n} a_{i,k} - a_{m,k} \sum_{i=1}^{n} a_{i,m} > 0. \tag{3.13}$$

Again from $\left(\sum_{i=1}^{n} a_{i,j} - a_{m,j}\right) > 1$, for $j = 1, 2, \ldots, n$, and from the maximum property

$$-\frac{a_{m,k}}{\sum_{i=1}^{n} a_{i,k} - a_{m,k}} \geq -\frac{a_{m,j}}{\sum_{i=1}^{n} a_{i,j} - a_{m,j}},$$

it follows that

$$-a_{m,k} \left(\sum_{i=1}^{n} a_{i,j} - a_{m,j} \right) \geq -a_{m,j} \left(\sum_{i=1}^{n} a_{i,k} - a_{m,k} \right)$$

or, after simplification,

$$a_{m,j} \sum_{i=1}^{n} a_{i,k} - a_{m,k} \sum_{i=1}^{n} a_{i,j} \geq 0. \tag{3.14}$$

The inequalities (3.13) and (3.14) imply that $0 \leq \overline{a}_{m,j}$, $j = 1, 2, \ldots, n$, or $\mathbf{0} \leq \overline{A}_m$. What remains is to show that $\overline{a}_{m,j} \leq a_{m,j}$, for $j = 1, 2, \ldots, n$.

From the inequality (3.13) one has

$$\sum_{i=1}^{n} a_{i,j} - a_{m,j} \sum_{i=1}^{n} a_{i,m} > 0.$$

Then multiplying both sides of the inequality by $a_{m,k} > 0$ one gets

$$a_{m,k} \left(\sum_{i=1}^{n} a_{i,j} - a_{m,j} \sum_{i=1}^{n} a_{i,m} \right) > 0$$

or

$$-a_{m,k} \sum_{i=1}^{n} a_{i,j} < -a_{m,k} a_{m,j} \sum_{i=1}^{n} a_{i,m}. \tag{3.15}$$

Now, adding $a_{m,j} \sum_{i=1}^{n} a_{i,k}$ to both sides of the inequality (3.15) , one obtains

$$a_{m,j} \sum_{i=1}^{n} a_{i,k} - a_{m,k} \sum_{i=1}^{n} a_{i,j} < a_{m,j} \left(\sum_{i=1}^{n} a_{i,k} - a_{m,k} \sum_{i=1}^{n} a_{i,m} \right)$$

and, finally dividing both sides by $\left(\sum_{i=1}^{n} a_{i,k} - a_{m,k} \sum_{i=1}^{n} a_{i,m}\right) > 0$ inequality (3.15) turns into

$$0 \leq \overline{a}_{m,j} = \frac{a_{m,j} \sum\limits_{i=1}^{n} a_{i,k} - a_{m,k} \sum\limits_{i=1}^{n} a_{i,j}}{\sum\limits_{i=1}^{n} a_{i,k} - a_{m,k} \sum\limits_{i=1}^{n} a_{i,m}} < a_{m,j}, \; j = 1, 2, \ldots, n,$$

which implies that $\overline{A}_m < A$. So, combining this with the previous inequality, $\mathbf{0} \leq \overline{A}_m$ it follows that $\mathbf{0} \leq \overline{A}_m \leq A$. □

It is clear that Lemma 3.2.1 holds for $m = n_1, n_2, \ldots, n_l$, where $l \in \{1, 2, \ldots, n\}$. Therefore, we can state the following.

Theorem 3.2.2 *Let $A = [a_{i,j}]$ be a SCDD positive matrix with unit diagonal entries and let $\overline{A}_C = \left[\overline{a}_{i,j}\right] = \overline{D}_C (I + P_C) A$, where P_C is given by*

$$P_C = \begin{bmatrix} 0 & w_1 & \cdots & \cdots & w_1 \\ w_2 & 0 & \cdots & \cdots & w_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{n-1} & w_{n-1} & \cdots & 0 & w_{n-1} \\ w_n & w_n & \cdots & w_n & 0 \end{bmatrix},$$

and $\overline{D}_C = diag\Big(\left(1 + w_1 \left(\sum_{i=1}^{n} a_{i,1} - 1\right)\right)^{-1}, \ldots, \left(1 + w_n \left(\sum_{i=1}^{n} a_{i,n} - 1\right)\right)^{-1} \Big)$. Moreover, let $w_l = w_{l,k_l} = \max\limits_{1 \leq j \leq n,\, j \neq l} \left\{ w_{l,j} \right\}$ for $l = 1, 2, \ldots, n$, where

$$w_{l,j} = -a_{l,j} / \left(\sum\nolimits_{i=1}^{n} a_{i,j} - a_{l,j} \right).$$

Then $0 \le \overline{a}_{i,j} < a_{i,j}$ *for* $i, j = 1, 2, \ldots, n$ *with* $i \ne j$ *and* $\mathbf{0} \le \overline{A}_C \le A$.

Let $A > \mathbf{0}$ be a SCDD matrix with unit diagonal entries and let $\overline{\overline{A}}_m = [\overline{\overline{a}}_{i,j}] = \overline{\overline{D}}_m(I + P_m)A$, where $\overline{\overline{D}}_m$ =diag$\Big(1, \ldots, 1, (1 + w_m\, (\sum_{i=1}^{n} a_{i,m} - 1))^{-1}, 1, \ldots, 1\Big)$. Then, we have the following.

Lemma 3.2.3 *If* $w_m = w_{m,k} = \max\limits_{1 \le j \le n,\ j \ne m} \left\{ w_{m,j} \right\}$, *where*

$$w_{m,j} = -2a_{m,j} / \left(\sum\nolimits_{i=1}^{n} a_{i,j} + a_{m,j} \sum\nolimits_{i=1}^{n} a_{i,m} - 2a_{m,j} \right), \qquad (3.16)$$

then $-a_{m,j} \le \overline{\overline{a}}_{m,j} \le a_{m,j}$ *for* $j = 1, 2, \ldots, n$, $j \ne m$, *and* $-A \le \overline{\overline{A}}_m \le A$.

Proof. Consider the inequality (3.13) or $(\sum_{i=1}^{n} a_{i,j} - a_{m,j} \sum_{i=1}^{n} a_{i,m}) > 0$ for $j = k$. Then adding $(-a_{m,k} \sum_{i=1}^{n} a_{i,m} + 2a_{m,k})$ to both sides of the inequality one obtains

$$-2a_{m,k} \left(\sum_{i=1}^{n} a_{i,m} - 1 \right) > -\left(\sum_{i=1}^{n} a_{i,k} + a_{m,k} \sum_{i=1}^{n} a_{i,m} - 2a_{m,k} \right) > 0$$

and dividing by $(\sum_{i=1}^{n} a_{i,k} + a_{m,k} \sum_{i=1}^{n} a_{i,m} - 2a_{m,k}) > 0$ gets the inequality (3.2).

We have

$$\overline{\overline{a}}_{m,j} = \frac{w_m \left(\sum\limits_{i=1}^{n} a_{i,j} - a_{m,j} \right) + a_{m,j}}{1 + w_m \left(\sum\limits_{i=1}^{n} a_{i,m} - 1 \right)}, \text{ for } j = 1, 2, \ldots, n.$$

Then taking $w_m = w_{m,k} = -2a_{m,k} / \left(a_{m,k} \sum_{i=1}^{n} a_{i,m} + \sum_{i=1}^{n} a_{i,k} - 2a_{m,k} \right)$ we obtain

$$\overline{\overline{a}}_{m,j} = \frac{-2a_{m,k} \left(\sum\limits_{i=1}^{n} a_{i,j} - a_{m,j} \right) + a_{m,j} \left(a_{m,k} \sum\limits_{i=1}^{n} a_{i,m} + \sum\limits_{i=1}^{n} a_{i,k} - 2a_{m,k} \right)}{\left(a_{m,k} \sum\limits_{i=1}^{n} a_{i,m} + \sum\limits_{i=1}^{n} a_{i,k} - 2a_{m,k} \right) - 2a_{m,k} \left(\sum\limits_{i=1}^{n} a_{i,m} - 1 \right)}, \text{ for } j = 1, 2, \ldots, n. \qquad (3.17)$$

or simplifying (3.17) we get

$$\overline{\overline{a}}_{m,j} = \frac{-2a_{m,k} \sum\limits_{i=1}^{n} a_{i,j} + a_{m,j} \left(\sum\limits_{i=1}^{n} a_{i,k} + a_{m,k} \sum\limits_{i=1}^{n} a_{i,m} \right)}{\sum\limits_{i=1}^{n} a_{i,k} - a_{m,k} \sum\limits_{i=1}^{n} a_{i,m}}, \text{ for } j = 1, 2, \ldots, n.$$

For $j = m$ one has

$$\begin{aligned}\overline{\overline{a}}_{m,m} &= \frac{-2a_{m,k}\sum\limits_{i=1}^{n} a_{i,m} + \left(\sum\limits_{i=1}^{n} a_{i,k} + a_{m,k}\sum\limits_{i=1}^{n} a_{i,m}\right)}{\sum\limits_{i=1}^{n} a_{i,k} - a_{m,k}\sum\limits_{i=1}^{n} a_{i,m}} \\ &= \frac{\sum\limits_{i=1}^{n} a_{i,k} - a_{m,k}\sum\limits_{i=1}^{n} a_{i,m}}{\sum\limits_{i=1}^{n} a_{i,k} - a_{m,k}\sum\limits_{i=1}^{n} a_{i,m}} = 1 > 0.\end{aligned}$$

Now, from (3.13) for $j = 1, 2, \ldots, n$, $j \neq m$, we have

$$0 < a_{m,j}\sum_{i=1}^{n} a_{i,m} < \sum_{i=1}^{n} a_{i,j},$$

which yields

$$0 > -2a_{m,k}a_{m,j}\sum_{i=1}^{n} a_{i,m} > -2a_{m,k}\sum_{i=1}^{n} a_{i,j},$$

or

$$-a_{m,k}a_{m,j}\sum_{i=1}^{n} a_{i,m} - a_{m,k}a_{m,j}\sum_{i=1}^{n} a_{i,m} > -2a_{m,k}\sum_{i=1}^{n} a_{i,j}.$$

Then, adding $a_{m,j}\sum_{i=1}^{n} a_{i,k}$ to both sides of the last inequality we obtain

$$a_{m,j}\sum_{i=1}^{n} a_{i,k} - a_{m,k}a_{m,j}\sum_{i=1}^{n} a_{i,m} - a_{m,k}a_{m,j}\sum_{i=1}^{n} a_{i,m} > -2a_{m,k}\sum_{i=1}^{n} a_{i,j} + a_{m,j}\sum_{i=1}^{n} a_{i,k},$$

or

$$a_{m,j}\left(\sum_{i=1}^{n} a_{i,k} - a_{m,k}\sum_{i=1}^{n} a_{i,m}\right) > -2a_{m,k}\sum_{i=1}^{n} a_{i,j} + a_{m,j}\left(\sum_{i=1}^{n} a_{i,k} + a_{m,k}\sum_{i=1}^{n} a_{i,m}\right).$$

Finally, dividing all sides by $(\sum_{i=1}^{n} a_{i,k} - a_{m,k}\sum_{i=1}^{n} a_{i,m}) > 0$ we get

$$\overline{\overline{a}}_{m,j} = \frac{-2a_{m,k}\sum\limits_{i=1}^{n} a_{i,j} + a_{m,j}\left(\sum_{i=1}^{n} a_{i,k} + a_{m,k}\sum\limits_{i=1}^{n} a_{i,m}\right)}{\left(\sum\limits_{i=1}^{n} a_{i,k} - a_{m,k}\sum\limits_{i=1}^{n} a_{i,m}\right)} < a_{m,j}. \tag{3.18}$$

From (3.16) for $j = 1, 2, \ldots, n$, $j \neq m$, it follows that

$$-\frac{2a_{m,k}}{\sum\limits_{i=1}^{n} a_{i,k} + a_{m,k}\sum\limits_{i=1}^{n} a_{i,m} - 2a_{m,k}} \geq \frac{-2a_{m,j}}{\sum\limits_{i=1}^{n} a_{i,j} + a_{m,j}\sum\limits_{i=1}^{n} a_{i,m} - 2a_{m,j}}.$$

Now, $\left(\sum_{i=1}^{n} a_{i,j} + a_{m,j}\sum_{i=1}^{n} a_{i,m} - 2a_{m,j}\right) = a_{m,j}(\sum_{i=1}^{n} a_{i,m} - 1) + \left(\sum_{i=1}^{n} a_{i,j} - a_{m,j}\right) > 0$, since $(\sum_{i=1}^{n} a_{i,m} - 1) > 0$ and $\left(\sum_{i=1}^{n} a_{i,j} - a_{m,j}\right) > 0$, $j = 1, 2, \ldots, n$, $j \neq m$. So, one has

$$-2a_{m,k}\left(\sum_{i=1}^{n} a_{i,j} + a_{m,j}\sum_{i=1}^{n} a_{i,m} - 2a_{m,j}\right) \geq -2a_{m,j}\left(\sum_{i=1}^{n} a_{i,k} + a_{m,k}\sum_{i=1}^{n} a_{i,m} - 2a_{m,k}\right)$$

or, expanding and simplifying one obtains

$$a_{m,k}\sum_{i=1}^{n}a_{i,j}-a_{m,j}\sum_{i=1}^{n}a_{i,k}\leq 0$$

or

$$-2a_{m,k}\sum_{i=1}^{n}a_{i,j}+a_{m,j}\sum_{i=1}^{n}a_{i,k}\geq -a_{m,j}\sum_{i=1}^{n}a_{i,k}. \tag{3.19}$$

Then, adding $a_{m,j}a_{m,k}\sum_{i=1}^{n}a_{i,m}$ to both sides of the inequality (3.19) one gets

$$-2a_{m,k}\sum_{i=1}^{n}a_{i,j}+a_{m,j}a_{m,k}\sum_{i=1}^{n}a_{i,m}+a_{m,j}\sum_{i=1}^{n}a_{i,k}\geq -a_{m,j}\sum_{i=1}^{n}a_{i,k}+a_{m,j}a_{m,k}\sum_{i=1}^{n}a_{i,m}$$

or

$$-2a_{m,k}\sum_{i=1}^{n}a_{i,j}+a_{m,j}\left(a_{m,k}\sum_{i=1}^{n}a_{i,m}+\sum_{i=1}^{n}a_{i,k}\right)\geq -a_{m,j}\left(\sum_{i=1}^{n}a_{i,k}-a_{m,k}\sum_{i=1}^{n}a_{i,m}\right).$$

Finally, dividing all sides by $(\sum_{i=1}^{n}a_{i,k}-a_{m,k}\sum_{i=1}^{n}a_{i,m})>0$, we obtain

$$\overline{\overline{a}}_{m,j}=\frac{-2a_{m,k}\sum\limits_{i=1}^{n}a_{i,j}+a_{m,j}\left(a_{m,k}\sum_{i=1}^{n}a_{i,m}+\sum\limits_{i=1}^{n}a_{i,k}\right)}{\left(\sum\limits_{i=1}^{n}a_{i,k}-a_{m,k}\sum_{i=1}^{n}a_{i,m}\right)}\geq -a_{m,j}. \tag{3.20}$$

Then, the inequality (3.18) together with (3.20) yields $-a_{m,j}\leq\overline{\overline{a}}_{m,j}\leq a_{m,j}$ for $j=1,2,\ldots,n$, $j\neq m$, and hence $-A\leq\overline{\overline{A}}_m\leq A$. So, the proof is completed. □

Since Lemma 3.2.3 holds for $m=n_1,n_2,\ldots,n_l$, where $l\in\{1,2,\ldots,n\}$. we can state the following.

Theorem 3.2.4 *Let $A=\left[a_{i,j}\right]$ be a SCDD positive matrix with unit diagonal entries and let $\overline{\overline{A}}_C=\left[\overline{\overline{a}}_{i,j}\right]=\overline{\overline{D}}_C\left(I+P_C\right)A$, where P_C is given by*

$$P_C=\begin{bmatrix}0 & w_1 & \cdots & \cdots & w_1\\ w_2 & 0 & \cdots & \cdots & w_2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ w_{n-1} & w_{n-1} & \cdots & 0 & w_{n-1}\\ w_n & w_n & \cdots & w_n & 0\end{bmatrix}$$

and $\overline{\overline{D}}_C=diag\Big(\left(1+w_1\left(\sum_{i=1}^{n}a_{i,1}-1\right)\right)^{-1},\ldots,\left(1+w_n\left(\sum_{i=1}^{n}a_{i,n}-1\right)\right)^{-1}\Big)$. Moreover, let $w_l=w_{l,k_l}=\max\limits_{1\leq j\leq n,\ j\neq l}\left\{w_{l,j}\right\}$ for $l=1,2,\ldots,n$, where

$$w_{l,j}=-2a_{l,j}/\left(\sum_{i=1}^{n}a_{i,j}+a_{l,j}\sum_{i=1}^{n}a_{i,l}-a_{l,j}\right).$$

Then $-a_{i,j} \leq \overline{\overline{a}}_{i,j} < a_{i,j}$ *for* $i, j = 1, 2, \ldots, n$ *with* $i \neq j$ *and* $-A \leq \overline{\overline{A}}_C \leq A$.

Having constructed preconditioners for the linear systems where the coefficient matrices are SCDD $L-$matrices with negative off-diagonal entries and SCDD positive matrices, we can discuss the convergence of Jacobi and Gauss-Seidel iterations for preconditioned linear systems. For convenience we shall use $\overline{A}$ to denote preconditioned coefficient matrices $\overline{A}_m$, $\overline{A}_P$ or $\overline{A}_C$. Similarly, $\overline{\overline{A}}$ will be used to denote preconditioned coefficient matrices $\overline{\overline{A}}_m$, $\overline{\overline{A}}_P$ or $\overline{\overline{A}}_C$ and, new preconditioning methods that lead to $\overline{A}$ and $\overline{\overline{A}}$ will be referred as Type-I and Type-II preconditionings, respectively.

3.3 Comparison Theorems For Convergence

For a simple introduction to the topic, we consider the linear system $Ax = b$. Let A, which is a SCDD L-matrix with unit diagonal entries, be given by

$$A = \begin{bmatrix} 1 & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & 1 & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & 1 & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & 1 \end{bmatrix} \tag{A-1}$$

and consider the decomposition of the form $A = I - L - U$ where I is the $n \times n$ identity matrix,

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ -a_{2,1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n-1,1} & -a_{n-1,2} & \cdots & 0 & 0 \\ -a_{n,1} & -a_{n,2} & \cdots & -a_{n,n-1} & 0 \end{bmatrix}$$

and

$$U = \begin{bmatrix} 0 & -a_{1,2} & \cdots & -a_{1,n-1} & -a_{1,n} \\ 0 & 0 & \cdots & -a_{2,n-1} & -a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{n-1,n} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

It is clear that $L \geq \mathbf{0}$ and $U \geq \mathbf{0}$. So,

$$I - L = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ a_{2,1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & 1 & 0 \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & 1 \end{bmatrix}$$

is also a SCDD L-matrix. Consider the augmented matrix $[I - L|I]$ which will be used in finding $(I - L)^{-1}$ by using Gaussian elimination on $[I - L|I]$. Recall that each elementary row operation on $[I-L|I]$ corresponds to pre-multiplication of $[I-L|I]$ by E, where E is the elementary matrix obtained by applying the same elementary row operation on I. Consider the augmented matrix $[I - L|I]$:

$$[I - L|I] = \left[\begin{array}{ccccc|ccccc} 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ a_{2,1} & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & 1 & 0 & 0 & 0 & \cdots & 1 & 0 \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & 1 & 0 & 0 & \cdots & 0 & 1 \end{array}\right]$$

Let E_1 be the product of the elementary matrices used in the elimination process in the 1st column of $(I - L)$. Since $a_{j,1} \leq 0,\ j = 2, 3, ..., n$ (due to $L-$matrix structure of A), the 1st column elements of E_1 are nonnegative and hence, $E_1 \geq \mathbf{0}$. Since

$$E_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -a_{2,1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n-1,1} & 0 & \cdots & 1 & 0 \\ -a_{n,1} & 0 & \cdots & 0 & 1 \end{bmatrix},$$

we get

$$E_1[I - L|I] = \left[\begin{array}{ccccc|ccccc} 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & -a_{2,1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n-1,2} & \cdots & 1 & 0 & -a_{n-1,1} & 0 & \cdots & 1 & 0 \\ 0 & a_{n,2} & \cdots & a_{n,n-1} & 1 & -a_{n,1} & 0 & \cdots & 0 & 1 \end{array}\right]$$

After the 1st column elimination, the left side of the resultant augmented matrix is an $L-$matrix, while the right side is nonnegative, that is $E_1 I = E_1 \geq \mathbf{0}$. So, if E_i is the product of the elementary matrices used in the elimination process in the ith column, then it is clear that $E_i \geq \mathbf{0}$, $i = 1, 2, ..., n-1$. This implies that $(I-L)^{-1} = E_{n-1}E_{n-2}\dots E_2 E_1 I \geq \mathbf{0}$. That is, $(I-L)^{-1}$ is the product of nonsingular and nonnegative matrices $E_1, E_2, \dots, E_{n-2}$ and E_{n-1}. Since $L \geq \mathbf{0}, U \geq \mathbf{0}$ and $(I-L)^{-1} \geq \mathbf{0}$, Jacobi iteration matrix $T_J = L + U \geq \mathbf{0}$ and Gauss-Seidel iteration matrix $T_G = (I-L)^{-1} U \geq \mathbf{0}$.

Now, we can consider the iteration matrices of Jacobi and Gauss-Seidel methods for the preconditioned linear systems $\overline{A}x = \overline{b}$ and $\overline{\overline{A}}x = \overline{\overline{b}}$, which we have introduced in the previous sections.

Case (I) The Preconditioned Linear System $\overline{A}x = \overline{b}$, where $\overline{A} = (\overline{D})^{-1}PA$, $\overline{b} = (\overline{D})^{-1}Pb$ and A is a SCDD $L-$matrix.

Here we consider the linear system which is obtained from Type-I preconditioning on A, whether based on a single row, on a limited number of rows or on all rows of A. Let $\overline{A}$, which is again a SCDD L-matrix with unit diagonal entries, be given by

$$\overline{A} = \begin{bmatrix} 1 & \overline{a}_{1,2} & \cdots & \overline{a}_{1,n-1} & \overline{a}_{1,n} \\ \overline{a}_{2,1} & 1 & \cdots & \overline{a}_{2,n-1} & \overline{a}_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{a}_{n-1,1} & \overline{a}_{n-1,2} & \cdots & 1 & \overline{a}_{n-1,n} \\ \overline{a}_{n,1} & \overline{a}_{n,2} & \cdots & \overline{a}_{n,n-1} & 1 \end{bmatrix}.$$

Since $\overline{A}$ is obtained from A by Type-I preconditioning, we have $a_{i,j} \leq \overline{a}_{i,j} \leq 0$ for $i, j = 1, 2, ..., n$, $i \neq j$. Consider the decomposition of $\overline{A}$ in the form $\overline{A} = I - \overline{L} - \overline{U}$,

$$\overline{L} = -\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \overline{a}_{2,1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{a}_{n-1,1} & \overline{a}_{n-1,2} & \cdots & 0 & 0 \\ \overline{a}_{n,1} & \overline{a}_{n,2} & \cdots & \overline{a}_{n,n-1} & 0 \end{bmatrix}, \quad \overline{U} = -\begin{bmatrix} 0 & \overline{a}_{1,2} & \cdots & \overline{a}_{1,n-1} & \overline{a}_{1,n} \\ 0 & 0 & \cdots & \overline{a}_{2,n-1} & \overline{a}_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \overline{a}_{n-1,n} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

It is clear that $\overline{L} \geq \mathbf{0}$ and $\overline{U} \geq \mathbf{0}$. So, the matrix $I - \overline{L}$ given by

$$I-\overline{L}=\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \overline{a}_{2,1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{a}_{n-1,1} & \overline{a}_{n-1,2} & \cdots & 1 & 0 \\ \overline{a}_{n,1} & \overline{a}_{n,2} & \cdots & \overline{a}_{n,n-1} & 1 \end{bmatrix}$$

is a SCDD L-matrix. Now, consider the augmented matrix $[I-\overline{L}|I]$ which will be used in finding $\left(I-\overline{L}\right)^{-1}$:

$$[I-\overline{L}|I]=\left[\begin{array}{ccccc|ccccc} 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \overline{a}_{2,1} & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{n-1,1} & \overline{a}_{n-1,2} & \cdots & 1 & 0 & 0 & 0 & \cdots & 1 & 0 \\ \overline{a}_{n,1} & \overline{a}_{n,2} & \cdots & \overline{a}_{n,n-1} & 1 & 0 & 0 & \cdots & 0 & 1 \end{array}\right].$$

If $\overline{E}_i, i=1,2,\ldots,n-1$ is the product of the elementary matrices used in the elimination process in the ith column, then it is clear that $\overline{E}_i \geq \mathbf{0}, i=1,2,...,n-1$, since $a_{j,1} \leq 0,\ j=2,3,...,n$ (due to L–matrix structure of $\overline{A}$). This implies that $\left(I-\overline{L}\right)^{-1} = \overline{E}_{n-1}\overline{E}_{n-2}\ldots\overline{E}_2\overline{E}_1 I \geq \mathbf{0}$. That is, $\left(I-\overline{L}\right)^{-1}$ is the product of nonsingular nonnegative matrices $\overline{E}_1,\overline{E}_2,\ldots,\overline{E}_{n-2}$ and $\overline{E}_{n-1}$. Since $\overline{A}=I-\overline{L}-\overline{U}$ is a SCDD L–matrix, where $\overline{L}\geq\mathbf{0},\overline{U}\geq\mathbf{0}$, it follows that the Jacobi iteration matrix $\overline{T}_J=\overline{L}+\overline{U}\geq\mathbf{0}$ and the Gauss-Seidel iteration matrix $\overline{T}_G=\left(I-\overline{L}\right)^{-1}\overline{U}\geq\mathbf{0}$.

Now consider the coefficient matrices A and $\overline{A}$ of the linear system $Ax=b$ and the preconditioned linear system $\overline{A}x=\overline{b}$, where $\overline{A}=(\overline{D})^{-1}PA$ and $\overline{b}=(\overline{D})^{-1}Pb$, respectively.

Due to the structure of the Type-I preconditioning, i.e., $a_{i,j}\leq\overline{a}_{i,j}\leq 0$ for $i,j=1,2,...,n,\ i\neq j$, we have $\mathbf{0}\leq\overline{L}\leq L$ and $\mathbf{0}\leq\overline{U}\leq U$. Then,

$$\mathbf{0}\leq\overline{T}_J=\left(\overline{L}+\overline{U}\right)\leq T_J=(L+U),$$

and hence

$$\rho\left(\overline{T}_J\right)\leq\rho\left(T_J\right).$$

Again, since $a_{i,j}\leq\overline{a}_{i,j}\leq 0$ for $i,j=1,2,...,n,\ i\neq j$, we have $\mathbf{0}\leq\overline{E}_i\leq E_i,\ i=$

$1, 2, \ldots, n-1$, which implies that

$$\mathbf{0} \leq \left(I - \overline{L}\right)^{-1} = \overline{E}_{n-1}\overline{E}_{n-2} \ldots \overline{E}_2\overline{E}_1 I \leq (I - L)^{-1} = E_{n-1}E_{n-2} \ldots E_2E_1 I.$$

Moreover, $\mathbf{0} \leq \overline{L} \leq L$ and $\mathbf{0} \leq \overline{U} \leq U$ together with $\mathbf{0} \leq \left(I - \overline{L}\right)^{-1} \leq (I - L)^{-1}$ implies that

$$\mathbf{0} \leq \overline{T}_G = \left(I - \overline{L}\right)^{-1} \overline{U} \leq T_G = (I - L)^{-1} U,$$

and hence

$$\rho\left(\overline{T}_G\right) \leq \rho\left(T_G\right).$$

Case (II) The Preconditioned Linear System $\overline{\overline{A}}x = \overline{\overline{b}}$, where $\overline{\overline{A}} = (\overline{\overline{D}})^{-1} PA$, $\overline{\overline{b}} = (\overline{\overline{D}})^{-1} Pb$ and A is a SCDD $L-$matrix.

Here we consider the linear system which is obtained from Type-II preconditioning on A, whether based on a single row, on a limited number of rows or on all rows of A.

Let $\overline{\overline{A}}$, which is again SCDD with unit diagonal entries but not L**-**matrix anymore, be given by

$$\overline{\overline{A}} = \begin{bmatrix} 1 & \overline{\overline{a}}_{1,2} & \cdots & \overline{\overline{a}}_{1,n-1} & \overline{\overline{a}}_{1,n} \\ \overline{\overline{a}}_{2,1} & 1 & \cdots & \overline{\overline{a}}_{2,n-1} & \overline{\overline{a}}_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{\overline{a}}_{n-1,1} & \overline{\overline{a}}_{n-1,2} & \cdots & 1 & \overline{\overline{a}}_{n-1,n} \\ \overline{\overline{a}}_{n,1} & \overline{\overline{a}}_{n,2} & \cdots & \overline{\overline{a}}_{n,n-1} & 1 \end{bmatrix}.$$

Since $\overline{\overline{A}}$ is obtained from A by Type-II preconditioning, we have $a_{i,j} \leq \overline{\overline{a}}_{i,j} \leq -a_{i,j}$ for $i, j = 1, 2, ..., n$, $i \neq j$. This also means that $\overline{\overline{a}}_{i,j} \gtrless 0$, $i, j = 1, 2, ..., n$, $i \neq j$, that is the entries $\overline{\overline{a}}_{i,j}$ are not of constant sign. Consider the decomposition of $\overline{\overline{A}}$ in the form $\overline{\overline{A}} = I - \overline{\overline{L}} - \overline{\overline{U}}$, where I is the $n \times n$ identity matrix and

$$\overline{\overline{L}} = -\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \overline{\overline{a}}_{2,1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{\overline{a}}_{n-1,1} & \overline{\overline{a}}_{n-1,2} & \cdots & 0 & 0 \\ \overline{\overline{a}}_{n,1} & \overline{\overline{a}}_{n,2} & \cdots & \overline{\overline{a}}_{n,n-1} & 0 \end{bmatrix}, \quad \overline{\overline{U}} = -\begin{bmatrix} 0 & \overline{\overline{a}}_{1,2} & \cdots & \overline{\overline{a}}_{1,n-1} & \overline{\overline{a}}_{1,n} \\ 0 & 0 & \cdots & \overline{\overline{a}}_{2,n-1} & \overline{\overline{a}}_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \overline{\overline{a}}_{n-1,n} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Then,

$$I - \overline{\overline{L}} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \overline{\overline{a}}_{2,1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{\overline{a}}_{n-1,1} & \overline{\overline{a}}_{n-1,2} & \cdots & 1 & 0 \\ \overline{\overline{a}}_{n,1} & \overline{\overline{a}}_{n,2} & \cdots & \overline{\overline{a}}_{n,n-1} & 1 \end{bmatrix} \gtrless \mathbf{0},$$

i.e., $I - \overline{\overline{L}}$ is SCDD but not L-matrix anymore, as in Type-I preconditioning on SCDD $L-$matrices. In order to overcome this difficulty in the convergence analysis of Type-II preconditioning for SCDD $L-$matrices it will be useful to state the following.

Consider n dimensional real row vectors u, q and column vectors v, r with entries u_i, v_i, q_i and r_i, respectively. Moreover, let $u, v \geq \mathbf{0}$, that is $u_i, v_i \geq 0$, $i = 1, 2, \ldots, n$, and $-u_i \leq q_i \leq u_i$, $-v_i \leq r_i \leq v_i$, $i = 1, 2, \ldots, n$. Then

$$uv^T = u_1v_1 + u_2v_2 + \cdots + u_nv_n \geq 0$$

and

$$\begin{aligned} \left|qr^T\right| &= |q_1r_1 + q_2r_2 + \cdots + q_nr_n| \\ &\leq |q_1r_1| + |q_2r_2| + \cdots + |q_nr_n| \\ &\leq u_1v_1 + u_2v_2 + \cdots + u_nv_n = uv^T. \end{aligned}$$

So, using this property it can easily be seen that if U, Q, V and R are $n \times n$ real matrices, $U, V \geq \mathbf{0}$, and $-U \leq |Q| \leq U$, $-V \leq |R| \leq V$, then $-UV \leq |QR| \leq UV$.

Now, returning back to our proof, consider the augmented matrices $[I - L|I]$ and $[I - \overline{\overline{L}}|I]$:

$$[I - L|I] = \left[\begin{array}{ccccc|ccccc} 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ a_{2,1} & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & 1 & 0 & 0 & 0 & \cdots & 1 & 0 \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} & 1 & 0 & 0 & \cdots & 0 & 1 \end{array}\right]$$

and

$$[I-\overline{\overline{L}}|I] = \left[\begin{array}{ccccc|ccccc} 1 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \overline{\overline{a}}_{2,1} & 1 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{\overline{a}}_{n-1,1} & \overline{\overline{a}}_{n-1,2} & \cdots & 1 & 0 & 0 & 0 & \cdots & 1 & 0 \\ \overline{\overline{a}}_{n,1} & \overline{\overline{a}}_{n,2} & \cdots & \overline{\overline{a}}_{n,n-1} & 1 & 0 & 0 & \cdots & 0 & 1 \end{array}\right],$$

that will be used in finding $(I-L)^{-1}$ and $\left(I-\overline{\overline{L}}\right)^{-1}$, respectively. It is clear that $(I-L)^{-1} = E_{n-1}E_{n-2}\ldots E_1 I$ and $\left(I-\overline{\overline{L}}\right)^{-1} = \overline{\overline{E}}_{n-1}\overline{\overline{E}}_{n-2}\ldots\overline{\overline{E}}_1 I$. Since $-L \leq \mathbf{0}$ and $-L \leq |-\overline{\overline{L}}| \leq L$, it follows that $-E_i \leq |\overline{\overline{E}}_i| \leq E_i$ for $i = 1, 2, \ldots, n-1$. So,

$$-E_{n-1}E_{n-2}\ldots E_1 I \leq |\overline{\overline{E}}_{n-1}\overline{\overline{E}}_{n-2}\ldots\overline{\overline{E}}_1 I| \leq E_{n-1}E_{n-2}\ldots E_1 I$$

or

$$-(I-L)^{-1} \leq |\left(I-\overline{\overline{L}}\right)^{-1}| \leq (I-L).$$

Then, since $-U \leq |\overline{\overline{U}}| \leq U$, it follows that

$$-(I-L)^{-1}U \leq |\left(I-\overline{\overline{L}}\right)^{-1}\overline{\overline{U}}| \leq (I-L)^{-1}U$$

and hence,

$$\rho\left(\overline{\overline{T}}_G\right) \leq \rho(T_G),$$

which is the required result. Having obtained these result we can state the following.

Theorem 3.3.1 *Let $A = [a_{i,j}]$ be a SCDD $L-$matrix of the form (3.1) having at least one row of negative off-diagonal entries and let $\overline{A}_C$ and $\overline{\overline{A}}_C$ be the preconditioned coefficient matrices satisfying the hypothesis in Theorem 3.1.2 and Theorem 3.1.4, respectively. Then*

(a) $\rho\left(\overline{T}_J\right) \leq \rho(T_J)$,

(b) $\rho\left(\overline{\overline{T}}_J\right) \leq \rho(T_J)$,

(c) $\rho\left(\overline{T}_G\right) \leq \rho(T_G)$,

(d) $\rho\left(\overline{\overline{T}}_G\right) \leq \rho(T_G)$, *where $T_J, \overline{T}_J$ and $\overline{\overline{T}}_J$ are Jacobi iteration matrices associated with A, $\overline{A}_C$ and $\overline{\overline{A}}_C$, respectively and, $T_G, \overline{T}_G$ and $\overline{\overline{T}}_G$ are Gauss-Seidel iteration matrices associated with A, $\overline{A}_C$ and $\overline{\overline{A}}_C$, respectively.*

Before stating the convergence result about SCDD positive matrices, it will be useful to consider the following.

Remark 3.3.2 *Let A be the SCDD positive matrix given by*

$$A = \begin{bmatrix} 1 & 0.3 & 0.2 & 0.1 \\ 0.4 & 1 & 0.3 & 0.1 \\ 0.2 & 0.3 & 1 & 0.2 \\ 0.3 & 0.3 & 0.2 & 1 \end{bmatrix}.$$

Then Type-I preconditioning on row 1 yields

$$\overline{A} = \begin{bmatrix} 1 & 0.1901 & 0.0909 & 0 \\ 0.4 & 1 & 0.3 & 0.1 \\ 0.2 & 0.3 & 1 & 0.2 \\ 0.3 & 0.3 & 0.2 & 1 \end{bmatrix},$$

and a simple computation reveals that

$$(I - L)^{-1}\, U = \begin{bmatrix} 0 & -0.3000 & -0.2 & -0.1 \\ 0 & 0.12 & -0.22 & -0.06 \\ 0 & 0.024 & 0.106 & -0.162 \\ 0 & 0.0492 & 0.1048 & 0.0804 \end{bmatrix},$$

and

$$\left(I - \overline{L}\right)^{-1} \overline{U} = \begin{bmatrix} 0 & -0.1901 & -0.0909 & 0 \\ 0 & 0.0760 & -0.22636 & -0.1 \\ 0 & 0.0152 & 0.0973 & -0.17 \\ 0 & 0.0312 & 0.0869 & 0.064 \end{bmatrix}.$$

So, neither $|\overline{T_G}| \leq |T_G|$ nor $|\overline{T_G}| \geq |T_G|$. The same problem exists for Type-II preconditioning. For that reason, we can not provide a proof of that Type-I and Type-II preconditionings on SCDD positive matrices give smaller spectral radius, for the moment at least. On the other hand, for the Jacobi method, since $|\overline{T}_J| \leq |T_J|$ and $|\overline{\overline{T}}_J| \leq |T_J|$, we can state the following.

Theorem 3.3.3 *Let A be a SCDD positive matrix and let $\overline{A}_C$ and $\overline{\overline{A}}_C$ be the preconditioned coefficient matrices satisfying the hypothesis in Theorem 3.2.2 and Theorem 3.2.4, respectively. Then*

(a) $\rho\left(\overline{T}_J\right) \leq \rho\left(T_J\right)$,

(b) $\rho\left(\overline{\overline{T}}_J\right) \leq \rho\left(T_J\right)$,

where $T_J, \overline{T}_J$ and $\overline{\overline{T}}_J$ are Jacobi iteration matrices associated with the coefficient matrices A, $\overline{A}_C$ and $\overline{\overline{A}}_C$, respectively.

CHAPTER 4

NUMERICAL RESULTS

In this chapter we provide some numerical results on SCDD $L-$matrices, SCDD positive matrices, columnwise diagonally dominant (CDD) $L-$matrices, non-SCDD $L-$matrices and non-SCDD positive matrices.

For each of the preconditioning methods tested, either the spectral radius of the iteration matrix or the spectral radius of the iteration matrix and number of iterations are given. The spectral radius of the iteration matrix associated with any given preconditioner is denoted by $\rho(\cdot)$. In case of iterations, the termination criteria for the iterations is $\frac{\|Ax^{(k)}-b\|}{\|b\|} \leq 10^{-6}$. In the tables, *nit* is used to denote number of iterations. The subscripts in NM-I_i or NM-II_i indicates that preconditioning is performed on ith row, while NM-I_C or NM-II_C denotes the complete preconditioning. The notations Gu.([8]), Us.([51]), Ni.([41]) and Yu.([58]) denote the preconditioning methods introduced in ([8]), ([51]), ([41]) and ([58]), respectively. NM-I and NM-II represents Type-I and Type-II preconditionings, respectively, which we have introduced.

In the first part of this chapter, using some tests matrices existing in the literature, such as SCDD $L-$matrices with negative off-diagonal entries, SCDD $L-$matrices, SCDD positive matrices, non-SCDD $L-$matrices, non-SCDD positive matrices and CDD $L-$matrices, we compare the spectral radii of iteration matrices for Jabobi and Gauss-Seidel iterative methods for some of the preconditioners from the literature and for the preconditioners we have introduced.

In the second part, using some test matrices from the literature, we considered the spectral radii and iteration numbers for Jacobi and Gauss-Seidel iterative methods for the preconditioners mentioned above.

4.1 Comparison of Spectral Radii

Example 4.1.1 *The matrix A ([9]) is given by*

$$A = \begin{bmatrix} 1 & -0.0058 & -0.19350 & -0.25471 & -0.03885 \\ -0.28424 & 1 & -0.16748 & -0.21780 & -0.21577 \\ -0.24764 & -0.26973 & 1 & -0.18723 & -0.08949 \\ -0.13880 & -0.01165 & -0.25120 & 1 & -0.13236 \\ -0.25809 & -0.08162 & -0.13940 & -0.04890 & 1 \end{bmatrix}. \tag{4.1}$$

It is an SCDD L−matrix. The results are given in Table 4.1. According to the table, complete pivoting of Type-II gives the best results for Jacobi and the preconditioner $P_{RU} = (I + R + U)$ *([58]) gives the best result for the Gauss-Seidel method.*

Table 4.1: Spectral radii of iteration matrices

Jacobi	$\rho(\cdot)$	**Gauss-Seidel**	$\rho(\cdot)$
Unprec.	0.6291	Unprec.	0.3850
Gu.([8])	0.5848	Gu.([8])	0.2860
Us.([51])	0.4436	Us.([51])	0.1677
Ni.([41])	0.5555	Ni.([41])	0.2352
Yu.([58])	0.4272	Yu.([58])	0.1475
NM-I_2	0.5563	NM-I_2	0.3137
NM-I_3	0.5516	NM-I_3	0.3000
NM-I_C	0.4689	NM-I_C	0.2246
NM-II_2	0.4612	NM-II_2	0.2444
NM-II_3	0.4429	NM-II_3	0.2054
NM-II_C	0.3642	NM-II_C	0.1493

Example 4.1.2 *The matrix A ([21]) is given by*

$$A = \begin{bmatrix} 1 & -0.1 & -0.1 & -0.1 & -0.2 \\ -0.1 & 1 & -0.1 & -0.1 & -0.2 \\ -0.1 & -0.1 & 1 & -0.1 & -0.2 \\ -0.1 & -0.1 & -0.1 & 1 & -0.2 \\ -0.1 & -0.1 & -0.1 & -0.1 & 1 \end{bmatrix}. \tag{4.2}$$

It is a SCDD L−matrix. The results are given in Table 4.2. In this case, while the preconditioner $P_{RU} = (I + R + U)$ *([58]) gives the best result, except complete preconditioning of Type-I and Type-II, complete preconditioning of Type-I gives the best result for Jacobi and Gauss-Seidel methods.*

Table 4.2: Spectral radii of iteration matrices

Jacobi	$\rho(\cdot)$	**Gauss-Seidel**	$\rho(\cdot)$
Unprec.	0.4702	Unprec.	0.2434
Gu.([8])	0.4187	Gu.([8])	0.1497
Us.([51])	0.4436	Us.([51])	0.1677
Ni.([41])	0.2756	Ni.([41])	0.0635
Yu.([58])	0.2379	Yu.([58])	0.0508
NM-I_1	0.3983	NM-I_1	0.1675
NM-I_3	0.3983	NM-I_3	0.1827
NM-I_C	$3.774e-09$	NM-I_C	$3.478e-17$
NM-II_1	0.2805	NM-II_1	0.1056
NM-II_3	0.2805	NM-II_3	0.0845
NM-II_C	0.0203	NM-II_C	0.0771

Example 4.1.3 *The matrix A ([32]) is given by*

$$A = \begin{bmatrix} 1 & -0.2 & -0.3 & -0.2 & -0.2 \\ -0.1 & 1 & -0.2 & -0.3 & -0.1 \\ -0.2 & -0.3 & 1 & -0.1 & -0.2 \\ -0.2 & -0.1 & -0.3 & 1 & -0.3 \\ -0.3 & -0.2 & -0.1 & -0.3 & 1 \end{bmatrix}. \tag{4.3}$$

It is a SCDD L−matrix. The results are given in Table 4.3. In this case, both of the complete preconditioning of Type-I and Type-II outperforms all the others included in the table, but while complete preconditioning of Type-I gives better result for Jacobi method, complete preconditioning of Type-II gives better result for Gauss-Seidel method.

Table 4.3: Spectral radii of iteration matrices

Jacobi	$\rho(\cdot)$	**Gauss-Seidel**	$\rho(\cdot)$
Unprec.	0.8403	Unprec.	0.7129
Gu.([8])	0.8055	Gu.([8])	0.6123
Us.([51])	0.7252	Us.([51])	0.4847
Ni.([41])	0.7690	Ni.([41])	0.5446
Yu.([58])	0.6990	Yu.([58])	0.4424
NM-I_1	0.6985	NM-I_1	0.4943
NM-I_3	0.6559	NM-I_3	0.4592
NM-I_C	0.2732	NM-I_C	0.0781
NM-II_1	0.4505	NM-II_1	0.2362
NM-II_3	0.3889	NM-II_3	0.1806
NM-II_C	0.3396	NM-II_C	0.0741

Example 4.1.4 *The matrix A ([58]) is given by*

$$A = \begin{bmatrix} 1 & -0.1 & -0.2 & -0.1 & -0.2 & -0.3 \\ -0.3 & 1 & -0.1 & -0.2 & -0.2 & -0.1 \\ 0 & -0.1 & 1 & -0.4 & -0.1 & -0.2 \\ -0.1 & -0.3 & -0.2 & 1 & -0.1 & -0.2 \\ -0.3 & -0.2 & -0.3 & 0 & 1 & -0.1 \\ -0.2 & -0.1 & -0.1 & 0 & -0.1 & 1 \end{bmatrix}. \quad (4.4)$$

It is a SCDD L−matrix with some zero elements So the preconditioning will be effective if it is applied to the 1*st,* 2*nd and* 4*th rows. The results are given in Table 4.4.*

Table 4.4: Spectral radii of iteration matrices

Jacobi	$\rho(\cdot)$	**Gauss-Seidel**	$\rho(\cdot)$
Unprec.	0.8078	Unprec.	0.6666
Gu.([8])	0.7746	Gu.([8])	0.5734
Us.([51])	0.6655	Us.([51])	0.4083
Ni.([41])	0.7478	Ni.([41])	0.5178
Yu.([58])	0.6537	Yu.([58])	0.3828
NM-I_1	0.7371	NM-I_1	0.5644
NM-I_4	0.7543	NM-I_4	0.5768
NM-I_C	0.5787	NM-I_C	0.3537
NM-II_1	0.6531	NM-II_1	0.4559
NM-II_4	0.6915	NM-II_4	0.4656
NM-II_C	0.3376	NM-II_C	0.1104

Example 4.1.5 *The matrix A ([41]) is given by*

$$A = \begin{bmatrix} 1 & -0.2 & -0.1 & -0.2 & -0.1 & -0.1 \\ -0.1 & 1 & -0.2 & -0.2 & -0.1 & -0.1 \\ -0.2 & -0.2 & 1 & -0.1 & -0.2 & -0.1 \\ -0.2 & -0.1 & -0.3 & 1 & -0.1 & -0.1 \\ -0.2 & -0.2 & -0.1 & -0.2 & 1 & -0.1 \\ -0.3 & -0.1 & -0.2 & -0.1 & -0.2 & 1 \end{bmatrix}. \quad (4.5)$$

It is a CDD L−matrix. The results are given in Table 4.5. According to the values given in the table, both of the complete preconditioning of Type-I and Type-II give the best results.

Table 4.5: Spectral radii of iteration matrices

Jacobi	$\rho(\cdot)$	**Gauss-Seidel**	$\rho(\cdot)$
Unprec.	0.7740	Unprec.	0.6065
Gu.([8])	0.7397	Gu.([8])	0.5076
Us.([51])	0.6359	Us.([51])	0.3601
Ni.([41])	0.7098	Ni.([41])	0.4595
Yu.([58])	0.6202	Yu.([58])	0.3404
NM-I_1	0.7164	NM-I_1	0.5293
NM-I_6	0.7129	NM-I_6	0.5252
NM-I_C	0.5077	NM-I_C	0.2758
NM-II_1	0.6514	NM-II_1	0.4528
NM-II_6	0.6399	NM-II_6	0.4419
NM-II_C	0.3889	NM-II_C	0.0930

Example 4.1.6 *The matrix A ([8]) is given by*

$$A = \begin{bmatrix} 1 & -0.2 & -0.1 & -0.4 & -0.2 \\ -0.2 & 1 & -0.3 & -0.1 & -0.6 \\ -0.3 & -0.2 & 1 & -0.1 & -0.6 \\ -0.1 & -0.1 & -0.1 & 1 & -0.01 \\ -0.2 & -0.3 & -0.4 & -0.3 & 1 \end{bmatrix}. \tag{4.6}$$

A is a non-SCDD L−matrix due to the last column. The results are given in Table 4.6.

Table 4.6: Spectral radii of iteration matrices

Jacobi	$\rho(\cdot)$	**Gauss-Seidel**	$\rho(\cdot)$
Unprec.	0.9807	Unprec.	0.9611
Gu.([8])	0.9780	Gu.([8])	0.9505
Us.([51])	0.9607	Us.([51])	0.9127
Ni.([41])	0.9592	Ni.([41])	0.9122
Yu.([58])	0.9530	Yu.([58])	0.8988
NM-I_2	0.9674	NM-I_2	0.9339
NM-I_5	0.9316	NM-I_5	0.8585
NM-I_C	0.8889	NM-I_C	0.7862
NM-II_2	0.9536	NM-II_2	0.9055
NM-II_5	0.8769	NM-II_5	0.7326
NM-II_C	0.9686	NM-II_C	0.4059

From the table it is seen that, although the matrix under consideration is not SCDD, preconditioning on a single row may give the smaller spectral radius among the iterative methods mentioned above. For the Jacobi method, in the sense of complete preconditining, Type-I preconditioning gives better result than Type-II preconditioning. For the Gauss-Seidel method, complete preconditioning of Type-II gives the best result.

Example 4.1.7 *The matrix A ([8]) is given by*

$$A = \begin{bmatrix} 1 & -0.0089 & -0.1305 & -0.0679 & -0.0252 \\ -0.2891 & 1 & -0.4724 & -0.2938 & -0.3628 \\ -0.1424 & -0.3383 & 1 & -0.0972 & -0.0290 \\ -0.3454 & -0.3384 & -0.4843 & 1 & -0.2982 \\ -0.0363 & -0.1415 & -0.3680 & -0.1266 & 1 \end{bmatrix}. \quad (4.7)$$

Again A is a non-SCDD L−matrix, due to the 3rd column. The results are given in Table 4.7. The observations on the results are very similar to the one in the previous example, but complete preconditioning of Type-II gives better results than that of Type-I for both Jacobi and Gauss-Seidel methods.

Table 4.7: Spectral radii of iteration matrices

Jacobi	$\rho(\cdot)$	**Gauss-Seidel**	$\rho(\cdot)$
Unprec.	0.8407	Unprec.	0.6897
Gu.([8])	0.7966	Gu.([8])	0.5610
Us.([51])	0.7018	Us.([51])	0.4506
Ni.([41])	0.7735	Ni.([41])	0.5329
Yu.([58])	0.6865	Yu.([58])	0.4393
NM-I_2	0.7194	NM-I_2	0.5241
NM-I_3	0.8119	NM-I_3	0.6345
NM-I_C	0.7170	NM-I_C	0.4710
NM-II_2	0.5343	NM-II_2	0.3566
NM-II_3	0.7814	NM-II_3	0.5745
NM-II_C	0.6145	NM-II_C	0.3265

Example 4.1.8 *The matrix A is a* 5×5 *SCDD positive matrix given by*

$$A = \begin{bmatrix} 1 & 0.1612 & 0.0794 & 0.2683 & 0.2996 \\ 0.2350 & 1 & 0.2081 & 0.2855 & 0.0744 \\ 0.1073 & 0.1402 & 1 & 0.1044 & 0.0544 \\ 0.3214 & 0.2813 & 0.1893 & 1 & 0.2279 \\ 0.0117 & 0.2922 & 0.2746 & 0.2571 & 1 \end{bmatrix}, \quad (4.8)$$

which is obtained as follows: First, the off-diagonal entries of a matrix $H = [h_{i,j}]$ are taken as random numbers generated by MATLAB command `rand`*. Then, for $j = 1, 2, \ldots, 5$, each entry $h_{j,j}$ is taken as the sum of a random number generated by* `rand` *and the sum of the off-diagonal entries in the jth column. Finally, $A = D^{-1}H$, where D is the diagonal part of H. The results are given in Table 4.8. For this matrix, Type-II complete preconditioner gives the smaller spectral radius for Jacobi method, while the preconditioner $P_{RU} = (I + R + U)$ ([58]) gives smallest spectral radius for the Gauss-Seidel method. In complete preconditioning, among the preconditioners, Type-I preconditioner gives the worst results.*

Table 4.8: Spectral radii of iteration matrices

Jacobi	$\rho(\cdot)$	**Gauss-Seidel**	$\rho(\cdot)$
Unprec.	0.7824	Unprec.	0.1647
Gu.([8])	0.3060	Gu.([8])	0.0900
Us.([51])	0.4083	Us.([51])	0.0572
Ni.([41])	0.2899	Ni.([41])	0.0876
Yu.([58])	0.2720	Yu.([58])	0.0442
NM-I_1	0.7279	NM-I_1	0.1447
NM-I_4	0.6154	NM-I_4	0.1132
NM-I_C	0.4617	NM-I_C	0.0621
NM-II_1	0.6661	NM-II_1	0.1219
NM-II_4	0.2784	NM-II_4	0.0903
NM-II_C	0.2645	NM-II_C	0.1144

Example 4.1.9 *The matrix A is a 15×15 SCDD positive matrix. It is generated in a similar way to the one in the previous example. The results are given in Table 4.9. For single row preconditioning, Type-I and Type-II preconditioners are not competitive with the other preconditioners in the table. Only Type-II complete preconditioner is better than that of Gunawardena et. all.'s preconditioner [8].*

Example 4.1.10 *The matrix A is a non-SCDD 5×5 positive matrix given by*

$$A = \begin{bmatrix} 1 & 0.1945 & 0.4160 & 0.2722 & 0.4854 \\ 0.1760 & 1 & 0.2986 & 0.2542 & 0.2774 \\ 0.0744 & 0.3802 & 1 & 0.2163 & 0.2807 \\ 0.3499 & 0.1631 & 0.1676 & 1 & 0.0816 \\ 0.4020 & 0.2943 & 0.1495 & 0.3358 & 1 \end{bmatrix}. \tag{4.9}$$

Table 4.9: Spectral radii of iteration matrices

Jacobi	$\rho(\cdot)$	**Gauss-Seidel**	$\rho(\cdot)$
Unprec.	0.9277	Unprec.	0.1995
Gu.([8])	0.7971	Gu.([8])	0.1770
Us.([51])	0.4244	Us.([51])	0.1100
Ni.([41])	0.6706	Ni.([41])	0.1160
Yu.([58])	0.3559	Yu.([58])	0.950
NM-I_1	0.9215	NM-I_1	0.1985
NM-I_{10}	0.9218	NM-I_{10}	0.1984
NM-I_C	0.8105	NM-I_C	0.1743
NM-II_1	0.9151	NM-II_1	0.1974
NM-II_{10}	0.9157	NM-II_{10}	0.1973
NM-II_C	0.6919	NM-II_C	0.1489

In constructing A, first, the off-diagonal entries of a matrix $H = [h_{i,j}]$ are taken as random numbers generated by MATLAB command `rand`*. Then, for $j = 1, 2, \ldots, 5$, each entry $h_{j,j}$ is taken as the difference of the sum of the off-diagonal entries in the jth column and a random number generated by* `rand`*. Finally, $A = D^{-1}H$, where D is the diagonal part of H. The results are given in Table 4.10. For single row preconditioning, Type-I and Type-II preconditioners are not better, but comparable with the others. On the other hand, Type-II complete preconditioner is superior among the preconditioners in the list.*

Table 4.10: Spectral radii of iteration matrices

Jacobi	$\rho(\cdot)$	**Gauss-Seidel**	$\rho(\cdot)$
Unprec.	**1.0528**	Unprec.	0.28541
Gu.([8])	**1.0528**	Gu.([8])	**1.0528**
Us.([51])	0.5995	Us.([51])	0.2331
Ni.([41])	0.5653	Ni.([41])	0.2227
Yu.([58])	0.4392	Yu.([58])	0.1650
NM-I_1	0.9220	NM-I_C	0.2541
NM-I_3	0.9993	NM-I_3	0.2490
NM-I_C	0.5904	NM-I_C	0.1867
NM-II_1	0.7486	NM-II_1	0.2502
NM-II_3	0.9406	NM-II_3	0.2082
NM-II_C	0.2821	NM-II_C	0.1348

4.2 Comparison of Spectral Radii and Iteration Numbers

In this section we give some results about the spectral radii and iteration numbers for some test matrices. The results also clearly indicate the importance of the relation between the spectral radius of iteration matrices and number of iterations.

Example 4.2.1 *We consider $n \times n$ SCDD L−matrices A ([51]) of the form*

$$A = \begin{bmatrix} 1 & a & b & c & a & \cdots \\ c & 1 & a & b & \ddots & a \\ b & c & \ddots & \ddots & \ddots & c \\ a & \ddots & \ddots & 1 & a & b \\ c & \ddots & b & c & 1 & a \\ \cdots & c & a & b & c & 1 \end{bmatrix},$$

where $a = -p/n$, $b = -p/(n+1)$ and $c = -p/(n+2)$. For some values of n and p, the results about the spectral radii of the iteration matrices and iteration numbers are given in Tables 4.11-4.13.

Table 4.11: Spectral radii and iteration numbers for $n = 10, p = 1$

Jacobi	$\rho(\cdot)$	*nit*	**Gauss-Seidel**	$\rho(\cdot)$	*nit*
Unprec.	0.8227	71	Unprec.	0.6864	37
Gu.([8])	0.8053	64	Gu.([8])	0.6305	31
Us.([51])	0.7198	43	Us.([51])	0.4733	19
Ni.([41])	0.7878	58	Ni.([41])	0.6015	28
Yu.([58])	0.7099	41	Yu.([58])	0.4595	18
NM-I_1	0.7396	46	NM-I_1	0.5632	25
NM-I_5	0.7396	46	NM-I_5	0.5632	25
NM-I_C	0.0671	6	NM-I_C	0.0189	4
NM-II_1	0.6137	32	NM-II_1	0.4210	17
NM-II_5	0.6137	32	NM-II_5	0.4210	17
NM-II_C	0.6885	38	NM-II_C	0.1467	8

Table 4.12: Spectral radii and iteration numbers for $n = 10, p = 0.7$

Jacobi	$\rho(\cdot)$	*nit*	**Gauss-Seidel**	$\rho(\cdot)$	*nit*
Unprec.	0.5659	26	Unprec.	0.3599	14
Gu.([8])	0.5475	23	Gu.([8])	0.2923	12
Us.([51])	0.4074	16	Us.([51])	0.1419	8
Ni.([41])	0.5195	22	Ni.([41])	0.2626	11
Yu.([58])	0.3931	17	Yu.([58])	0.1334	7
NM-I_1	0.5179	22	NM-I_1	0.2994	12
NM-I_5	0.5179	22	NM-I_5	0.5179	22
NM-I_C	0.0481	5	NM-I_C	0.0123	4
NM-II_1	0.4426	18	NM-II_1	0.2255	10
NM-II_5	0.4426	18	NM-II_5	0.2255	10
NM-II_C	0.4797	19	NM-II_C	0.0973	7

Table 4.13: Spectral radii and iteration numbers for $n = 30, p = 1$

Jacobi	$\rho(\cdot)$	*nit*	**Gauss-Seidel**	$\rho(\cdot)$	*nit*
Unprec.	0.9361	210	Unprec.	0.8777	107
Gu.([8])	0.9340	203	Gu.([8])	0.8697	100
Us.([51])	0.8989	131	Us.([51])	0.7785	55
Ni.([41])	0.9319	196	Ni.([41])	0.8657	97
Yu.([58])	0.8977	129	Yu.([58])	0.7761	54
NM-I_5	0.9054	140	NM-I_5	0.8224	71
NM-I_{25}	0.9054	140	NM-I_{25}	0.8225	72
NM-I_C	0.0351	5	NM-I_C	0.0074	4
NM-II_5	0.8724	102	NM-II_5	0.7656	53
NM-II_{25}	0.8725	103	NM-II_{25}	0.7658	53
NM-II_C	0.8610	93	NM-II_C	0.1872	9

Based on the data in the tables, our newly developed Type-I and Type-II single row preconditioners are quite competitive with the ones listed in the tables. On the other hand, Type-I and Type-II complete row preconditioners surpasses all the other preconditioners in the list. In particular, Type-I complete preconditioner is the best one, in the sense of spectral radius and number of iterations.

CHAPTER 5

CONCLUSION

In this thesis we have introduced new preconditioners for some stationary iterative methods, namely, Jacobi and Gauss-Seidel methods, for the solution of systems of linear equations of the form $Ax = b$, where A is either a SCDD $L-$matrix or a SCDD positive matrix. Both types of new preconditioners, Type-I and Type-II, are developed using the simple fact that if T_1 and T_2 are $n \times n$ real matrices such that $|T_1| \leq T_2$, where the ordering is based on the ordering of the corresponding entries, then $\rho(T_1) \leq \rho(T_2)$, where $\rho(\cdot)$ denotes the spectral radius of the matrix under consideration.

Both types of new preconditioning we have developed can be performed on a single row, on a limited number of rows (*partial preconditioning*) or on all rows (*complete preconditioning*) of A. In case of preconditioning on more than one row, preconditioning on the rows are performed *simultaneously* and *independently*. That is, if a partial preconditioning is performed on rows $n_1, n_2, \ldots, n_m$, and if the other is performed on the rows $n_1, n_2, \ldots, n_l$, $1 \leq m < l \leq n$, then the associated preconditioned matrices can differ only in rows $n_{m+1}, n_{m+2}, \ldots, n_l$. Therefore, in studying the properties of the preconditioned matrices, we considered preconditioning on a single row, and then we generalized the result.

Let $A = [a_{i,j}]$ be a SCDD $L-$matrix or a SCDD positive matrix. Let $\overline{A} = [\overline{a}_{i,j}]$ and $\overline{\overline{A}} = [\overline{\overline{a}}_{i,j}]$ be the preconditioned coefficient matrices associated with Type-I and Type-II preconditioning, partially or completely, respectively. In Type-I preconditioning the aim is to obtain the preconditioned matrix $\overline{A}$ in such a way that off-diagonal entries of $\overline{A}$ are either 0 or $a_{i,j} \cdot \overline{a}_{i,j} > 0$ and $|\, \overline{a}_{i,j} \,| \leq |\, a_{i,j} \,|$, $i, j = 1, 2, \ldots, n$. On the other hand, in Type-II preconditioning we construct $\overline{\overline{A}}$ in such a way that off-diagonal entries of $\overline{\overline{A}}$

satisfy $|\overline{\overline{a}}_{i,j}| \leq |a_{i,j}|$, $i, j = 1, 2, \ldots, n$.

It has been shown that, if A is a SCDD $L-$matrix having at least one row of negative off-diagonal entries, then the spectral radii of Jacobi and Gauss-Seidel iteration matrices associated with Type-I and Type-II preconditioned systems are smaller than the ones associated with the unpreconditioned systems. On the other hand, when A is a SCDD positive matrix, it has been proved that the spectral radii of Jacobi iteration matrix associated with Type-I and Type-II preconditioned systems are smaller than the ones associated with the unpreconditioned systems. However, in case of Gauss-Seidel iterative method, where A is a SCDD positive matrix, an ordering between the iteration matrices for preconditioned and unpreconditioned linear systems does not exist in general, that is, $\left|\overline{T}_G\right| \leq |T_G|$ and $\left|\overline{\overline{T}}_G\right| \leq |T_G|$ do not hold in general, and as a result, no theoretical proof similar to the one for SCDD $L-$matrices is available for Gauss-Seidel iterative method for systems with SCDD positive matrices.

Numerical tests have been conducted to exemplify the theoretical results and to measure the effectiveness of newly developed preconditioners. In this context, comparisons with some of the preconditioners in the literature have been performed, the main criteria being the spectral radii and number of iterations. Considering the numerical test results, it is seen that single row Type-I or Type-II preconditioners for SCDD $L-$ matrices are quite competitive with the ones on which we performed some numerical tests, and in some of them they are the best. Moreover, Type-I or Type-II complete preconditioners give best performances in most of the test problems. However, performance of our Type-I and Type-II preconditioners for SCDD positive matrices are not good, in general, although they are competitive for some test problems. Finally, performances of new preconditioners for some CDD $L-$matrices and non-CDD $L-$matrices and, even for non-SCDD positive matrices deserve further theoretical studies.

REFERENCES

[1] R. Bagnara, *A unified proof for the convergence of Jacobi and Gauss-Seidel methods*, SIAM Review, **37**(1995), 93-97.

[2] R.L. Burden, J.D. Faires, *Numerical Analysis, Ninth Edition*, Brooks/Cole, 2011.

[3] G.H. Cheng, T.Z. Huang, X.Y. Cheng, *Preconditioned Gauss-Seidel type iterative method for solving linear systems*, Applied Mathematics and Mechanics (English Edition), **27**(2006),1275-1279.

[4] B.N. Datta, *Numerical Linear Algebra and Applications, Second Edition*, SIAM, 2010.

[5] V. Edalatpour, D. Hezari , D.K. Salkuyeh, *A generalization of the Gauss-Seidel iteration method for solving absolute value equations*, Applied Mathematics and Computation, **293**(2017), 156-167.

[6] D.J. Evans, M.M. Martins,M.E.Trigo, *The AOR iterative method for new preconditioned linear systems*, Journal of Computational and Applied Mathematics, **132**(2001), 461-466.

[7] A. Greenbaum, *Iterative Methods for Solving Linear Systems*, SIAM, 1997.

[8] A.D. Gunawardena, *Modified iterative methods for consistent linear systems, Linear Algebra and its Applications*, **154-156**(1991), 123-143.

[9] A. Hadjidimos, D. Noutsos, M. Tsoumas, *More on modifications and improvements of classical iterative schemes for $M-$matrices*, Linear Algebra and its Applications, **364**(2003), 253-279.

[10] R.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge University Press, 1991.

[11] T.Z. Huang, G.H. Cheng, X.Y. Cheng, *Modified SOR-type iterative method for $Z-$matrices*, Applied Mathematics and Computation, **175**(2006), 258-268.

[12] T.Z. Huang, G.H. Cheng, D.J. Evans, X.Y. Cheng, *AOR type iterations for solving preconditioned linear systems*, International Journal of Computer Mathematics, **82**(2005), 969-976.

[13] T.Z. Huang, D.J. Evans, *New estimates of the spectral radii of J, G-S, and SOR iterative matrices for a class of linear systems*, International Journal of Computer Mathematics, **84**(2007), 23-31.

[14] T.Z. Huang, X.Z. Wang, Y.D. Fu, *Improving Jacobi methods for nonnegative $H-$matrices linear systems*, Applied Mathematics and Computation, **186**(2007), 1542-1550.

[15] K.R. James, *Convergence of matrix iterations subject to diagonal dominance*, SIAM Journal of Numerical Analysis, **10**(1973), 478-484.

[16] Y. Jiang, L.Zou, *Convergence of the Gauss-Seidel iterative method*, Procedia Engineering, **15**(2011), 1647-1650.

[17] X.Q. Jin, Y.M. Wei, H.S. Tam, *Preconditioning technique for symmetric $M-$matrices*, CALCOLO, **42**(2005), 105-113.

[18] X.Q. Jin, *$M-$preconditioner for $M-$matrices*, Applied Mathematics and Computation, **172**(2006), 701-707.

[19] T. Kohno, H. Kotakemori , H. Niki , *Improving the modified Gauss-Seidel method for $Z-$ Matrices,Linear Algebra and its Applications,* **267**(1997), 113-123.

[20] T. Kohno, H. Niki, *A note on the preconditioner $P_m = (I + S_m)$*, Journal of Computational and Applied Mathematics, **225**(2009), 316-319.

[21] T. Kohno, H. Niki, *Letter to the Editor: A note on the preconditioned Gauss-Seidel (GS) method for linear systems*, Journal of Computational and Applied Mathematics, **233**(2010), 2413-2421.

[22] H. Kotakemori, K. Harada, M. Morimoto, H. Niki, *A comparison theorem for the iterative method with the preconditioner $(I+S_{\max})$*, Journal of Computational and Applied Mathematics, **145** (2002), 373-378.

[23] H. Kotakemori, H. Niki, N. Okamoto, *A generalization of the adaptive Gauss-Seidel method for $Z-$matrices*, International Journal of Computer Mathematics, **64**(1997), 317-326.

[24] H. Kotakemori, H. Niki, N. Okamoto, *Accelerated iterative method for $Z-$matrices*, Journal of Computational and Applied Mathematics, **75**(1996), 87-97.

[25] H. Kotakemori, H. Niki, N. Okamoto, *Convergence of a preconditioned iterative method for $H-$matrices*, Journal of Computational and Applied Mathematics, **83**(1997), 115-118.

[26] G. Lei, J. Yang, Y. Chen, J. Yang, J. Li, *Convergence analysis of the new splitting preconditioned SOR-type iterative methods for the linear system*, Numerical Methods for Partial Differential Equations, **30**(2014), 1716-1726.

[27] Y.T. Li, C.X. Li, S.L. Wu, *Improvements of preconditioned AOR iterative method for $L-$matrices*, Journal of Computational and Applied Mathematics, **206**(2007), 656-665.

[28] W. Li, W. Sun, *Modified Gauss-Seidel type methods and Jacobi type methods for $Z-$matrices*, Linear Algebra and its Applications, **317**(2000), 227-240.

[29] W. Li, *A note on the preconditioned Gauss-Seidel(GS) method for linear systems*, Journal of Computational and Applied Mathematics, **182**(2005), 81-90.

[30] Q. Liu, J. Huang, S. Zeng, *Convergence analysis of two preconditioned iterative methods for $M-$matrix linear systems*, Journal of Computational and Applied Mathematics, **281**(2015), 49-57.

[31] Q. Liu, G. Chen, J. Cai , *Convergence analysis of the preconditioned Gauss-Seidel method for $H-$matrices*, Computers and Mathematics with Applications, **56**(2008), 2048-2053.

[32] Q. Liu, G. Chen, *Erratum to : A note on the preconditioned Gauss-Seidel method for M-matrices* (J. Comput. Appl. Math. 219(1)(2008), 59-71), Journal of Computational and Applied Mathematics, **228**(2009), 498-502.

[33] H. Mahawar, V. Sarin, *Parallel iterative methods for dense linear systems in inductance extraction*, Parallel Computing, **29**(2003), 1219-1235.

[34] S.X. Miao, B. Zheng, *Convergence analysis of the modified Gauss-Seidel iterative method for $H-$matrices*, Applied Mathematics E-Notes, **12**(2012), 70-78.

[35] J. P. Milaszewicz, *Improving Jacobi and Gauss-Seidel iterations*, Linear Algebra and its Applications, **93**(1987), 161-170.

[36] M.E. Mokari-Bolhassan, T.N. Trick, *A new iterative algorithm for the solution of large scale systems*, presented at 28th Midwest Symposium On Circuits and Systems Louisville, Ky, 19-20 Aug. 1985.

[37] M. Morimoto, K. Harada, M. Sakakihara, H. Sawami, *The Gauss-Seidel iterative method with the preconditioning matrix* $(I + S + S_m)$, Japan J. Indust. Appl. Math., **21**(2004), 25-34.

[38] M. Morimoto, *Study on the preconditioners* $(I + S_m)$, Journal of Computational and Applied Mathematics, **234**(2010), 209-214.

[39] H.S. Najafi, S.A. Edalatpanah, A.H.R. Sheikhani, *Convergence analysis of modified iterative methods to solve linear systems*, Mediterranean Journal of Mathematics, **11**(2014), 1019-1032.

[40] H. Niki, T. Kohno, K. Abe, *An extended GS method for dense linear systems*, Journal of Computational and Applied Mathematics, **231**(2009), 177-186.

[41] H. Niki, T. Kohno, N. Morimoto, *The preconditioned Gauss-Seidel method faster than the SOR method*, Journal of Computational and Applied Mathematics, **219**(2008), 59-71.

[42] H. Niki , K. Harada, M. Morimoto, M. Sakakihara, *The survey of preconditioners used for accelerating the rate of convergence in the Gauss-Seidel method*, Journal of Computational and Applied Mathematics, **164-165**(2004), 587-600.

[43] D. Noutsos, M. Tzoumas, *On optimal improvements of classical iterative schemes for Z-matrices*, Journal of Computational and Applied Mathematics, **188**(2006), 89-106.

[44] A. Y. Özban, *On accelerated iterative methods for the solution of systems of linear equations*, International Journal of Computer Mathematics, **97**(2002), 765-773.

[45] Y. Saad, H.A.V. Vorst, *Iterative solution of linear systems in the 20th century*, Journal of Computational and Applied Mathematics, **123**(2000), 1-33.

[46] M. Sakakihara, M. Morimoto, H. Niki, N. Okamoto, *Preconditioned iterative methods for linear equations arising from the boundary element method* , International Journal of Computational Fluid Dynamics, **18**(2004), 625-630.

[47] H. Shen, X. Shao, Z. Huang, C. Li , *Preconditioned Gauss-Seidel iterative method for Z-matrices linear systems*, Bulletin of The Korean Mathematical Society, **48**(2011), 303-314.

[48] J. Stoer, R. Bulirsch, Introduction to Numerical Analysis, Springer-Verlag, 1980.

[49] P. Tarazaga, D. Cuellar, *Preconditioners generated by minimizing norms*, Computers and Mathematics with Applications, **57**(2009), 1305-1312.

[50] L.N. Trefethen, D. Bau, III, *Numerical Linear Algebra*, SIAM, 1997.

[51] M. Usui, H. Niki, T. Kohno, *Adaptive Gauss-Seidel method for linear systems*, Intern. J. Computer Math., **31**(1994), pp. 119-125.

[52] S. Venit, *The convergence of Jacobi and Gauss-Seidel iteration*, Mathematics Magazine, **48**(1975), 163-167.

[53] X. Z. Wang, T. Z. Huang, Y. D. Fu, *Preconditioned diagonally dominant property for linear systems with H-matrices*, Applied Mathematics E-Notes, **6**(2006), 235-243.

[54] Z. D. Wang, T. Z. Huang, *The upper Jacobi and upper Gauss-Seidel type iterative methods for preconditioned linear systems*, Applied Mathematics Letters, **19**(2006), 1029-1036.

[55] L. Wang, Y. Song, *Preconditioned AOR iterative methods for $M-$matrices*, Journal of Computational and Applied Mathematics, **226**(2009), 114-124.

[56] L. Wang, *On a class a row preconditioners for solving linear systems*, International Journal of Computer Mathematics, **83**(2006), 939-949.

[57] S.L. Wu, T.Z. Huang, *Convergence of the preconditioned AOR method for irreducible $L-$matrices*, Applied Mathematics and Computation, **212**(2009), 551-552.

[58] J.Y. Yuan, D.D. Zontini, *Comparison theorems of preconditioned Gauss-Seidel methods for $M-$matrices*, Applied Mathematics and Computation, **219**(2012), 1947-1957.

[59] J.H. Yun, S.W. Kim, *Convergence of the preconditioned AOR method for irreducible $L-$matrices*, Applied Mathematics and Computation, **201**(2008), 56-64.

[60] Y. Zhang, T.Z. Huang, X.P. Liu, T.X. Gu, *A class of preconditioners on the $(I + S(\alpha))$ type preconditioning matrices for solving linear systems*, Applied Mathematics and Computation, **189**(2007), 1737-1748.

[61] Y. Zhang, T.Z. Huang, *Modified iterative methods for nonnegative matrices and $M-$matrices linear systems*, Computers and Mathematics with Applications, **50**(2005), 1587-1602.

[62] B. Zheng, S.X. Miao, *Two new modified Gauss-Seidel methods for linear system with $M-$matrices*, Journal of Computational and Applied Mathematics, **233**(2009), 922-930.