AUTOMORPHISMS OF THE GRAPH OF CURVES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
ATILIM UNIVERSITY

BY

AMEL OMAR ELAMIN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS

MAY 2017

Approval of the Graduate School of Natural and Applied Sciences, Atılım University.

Prof. Dr. İbrahim Akman
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of **Master of Science in Mathematics Department, Atılım University**.

Prof. Dr. Tanıl Ergenç
Head of Department

This is to certify that we have read the thesis AUTOMORPHISMS OF THE GRAPH OF CURVES submitted by AMEL OMAR ELAMIN and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Ferihe Atalan Ozan
Supervisor

Examining Committee Members:

Assoc. Prof. Dr. Sinem ONARAN
Mathematics Department, Hacettepe University

Assist. Prof. Dr. S. Öykü YURTTAŞ
Mathematics Department, Dicle University

Assoc. Prof. Dr. Ferihe ATALAN OZAN
Mathematics Department, Atılım University

Date: May 23, 2017

I declare and guarantee that all data, knowledge and information in this document has been obtained, processed and presented in accordance with academic rules and ethical conduct. Based on these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name : AMEL OMAR ELAMIN

Signature :

ABSTRACT

AUTOMORPHISMS OF THE GRAPH OF CURVES

Elamin, Amel Omar

M.S., Department of Mathematics

Supervisor : Assoc. Prof. Dr. Ferihe Atalan Ozan

May 2017, 31 pages

In this thesis, we study the automorphisms of a certain graph of nonseparating curves and automorphisms of complexes of two-sided curves on surfaces. In Chapter 3, we deal with the work of P. S. Schaller on mapping class groups of hyperbolic surfaces and automorphism groups of graphs for orientable surfaces of genus $g \geq 1$ with n punctures. More precisely, it is shown that the automorphism group of the certain graph is isomorphic to the extended mapping class group of the orientable surface proved by Schaller. In the last chapter of this thesis, we study the automorphisms of the complexes of two-sided simple closed curves on nonorientable surfaces of genus g with n holes.

Keywords: complex of curves, mapping class group, surfaces

ÖZ

EĞRİ GRAFININ OTOMORFİZMALARI

Elamin, Amel Omar

Yüksek Lisans, Matematik Bölümü

Tez Yöneticisi : Doç. Dr. Ferihe Atalan Ozan

Mayıs 2017, 31 sayfa

Bu tezde, yüzeyler üzerindeki ayırmayan eğrilerin belli bir grafının otomorfizmalarını ve iki-taraflı eğrilerin komplekslerinin otomorfizmalarını çalışacağız. Üçüncü bölümde, cins sayısı $g \geq 1$ ve n delikli yönlendirilebilen yüzeyler için P. S. Schaller'ın hiperbolik yüzeylerin gönderim sınıf grupları ve grafların otomorfizma grupları üzerine yaptığı çalışmasını ele alacağız. Daha açık olarak, Schaller tarafından ispat edilen belli bir grafın otomorfizma grubunun yönlendirilebilen yüzeyin genişletilmiş gönderim sınıf grubuna izomorfik olduğu gösterilmektedir. Bu tezin son bölümünde, cins sayısı g ve n delikli yönlendirilemeyen yüzeyler üzerindeki iki-taraflı basit kapalı eğrilerin komplekslerinin otomorfizmalarını çalışacağız.

Anahtar Kelimeler: eğrilerin kompleksi, gönderim sınıf grubu, yüzeyler

To my father, my mother, my husband and my sons (Yousef and Mohammed)

ACKNOWLEDGMENTS

First of all, I am thankful to **Allah** for all the gifts has given to me.

I shall also thank **my country** because it has provided me with the financial support for my graduate studies.

I would like to thank my supervisor **Assoc. Prof. Dr. Ferihe Atalan Ozan** for her continuous assistance, guidance and encouragement throughout my master studies.

Furthermore, I want to thank **Assoc. Prof. Dr. Sinem Onaran, Assist. Prof. Dr. S. Öykü Yurttaş and Assoc. Prof. Dr. Ferihe Atalan Ozan** as the members of examination commeittee for their curful reading this thesis and valuable comments.

Finally, I would like to express my deepest gratitude to **my mother Zakia, my father Omar, my sisters, my brothers, my husband Abd Alrahman and my sons** for their endless support and encouragement.

TABLE OF CONTENTS

LIST OF FIGURES

FIGURES

LIST OF SYMBOLS

Σ	:	A surface
S	:	An orientable surface
N	:	A nonorientable surface
$MCG(\Sigma)$	:	The mapping class group of Σ
$MCG^{*}(S)$	:	The extended mapping class group of S
$C(\Sigma)$	:	The complex of curves of Σ
$C_T(N)$	:	The complex of two-sided curves of N
$G_{ph}(S)$	:	The graph of an orientable surface S
$V(S)$	:	The set of nonseparating simple closed curves of the surface S
P	:	A partition
$\mathbf{P}$	:	A $P-S$ decomposition
C	:	A multicurve
ℓ	:	The maximal multicurve
$V(\alpha)$	:	The set of all curves on Σ distinct and disjoint from α
G	:	A group
$Aut(G)$	:	An automorphism group of G
$\kappa(N)$	:	The complexity of N

CHAPTER 1

INTRODUCTION

Let S be a compact, connected orientable surface of genus g with n punctures. The mapping class group of the surface S is defined as the group of all isotopy classes of orientation- preserving homeomorphisms $S \to S$. If we take account into orientation-reversing homeomorphisms, then we call this group the extended mapping class group. These groups are denoted by $MCG(S)$ and $MCG^*(S)$, respectively. If we consider a nonorientable surface denoted by N, the mapping class group $MCG(N)$ of N is the group of isotopy classes of all homeomorphisms $N \to N$.

Let $V(S)$ denote the set of isotopy classes of the simple closed curves of the surface S. If we take the sets of pairwise disjoint elements of $V(S)$, then some finite subsets of $V(S)$ have many significant applications. For example, a maximal set of pairwise disjoint elements $(3g - 3 + n)$ of $V(S)$ partitions S into pairs of pants. This gives parameters for the Teichmüller space of S. Another application is the complex of curves which is introduced by Harvey [8]. The complexes of curves are the fundamental geometric objects which have many applications. We are interested in the automorphism group of the curve complex of surfaces. The automorphism group of the curve complex of an orientable surface is isomorphic to the extended mapping class group. For a closed surface of genus $g \geq 2$, this fact is proved by Ivanov [10]. For genus $g = 2$, there exists the kernel which is $\mathbb{Z}_2$ generated by the hyperelliptic involution. Korkmaz [13] proved it for lower genus cases. Also, Luo [15] proved it using different methods in all cases independently. Atalan - Korkmaz [1] proved it for nonorientable surface of genus $g + n \geq 5$.

The goal of this thesis is to study automorphisms of a certain graph and the complex of two-sided curves on surfaces. Our investigation has two components. Following

Schaller's paper, first, we study the automorphism group of the certain graph $G_{ph}(S)$ of an orientable surface S. Second, we study the automorphism group of the subcomplex of two-sided curves $C_T(N)$ of a nonorientable surface N.

This thesis is organized as follows. In Chapter 2, we give some definitions, examples and fundamental concepts used in this work. In Chapter 3, we study the automorphism group of a certain graph denoted by $G_{ph}(S)$ of an orientable surface S of $g \geq 1$ with n punctures. The vertices of $G_{ph}(S)$ are nonseparating simple closed curves of the surface S. Two vertices are connected by an edge if they intersect exactly once. The automorphism group of this graph is isomorphic to the extended mapping class group of the surface S which is proved by Schaller [17]. In fact, he proved this fact for geodesic. Any surface S of negative Euler characteristic admits an hyperbolic metric, which is a Riemannian metric with Gaussian curvature equal to -1 at all points of the surface. It is well known that on any surface with hyperbolic metric every isotopy class of simple closed curve contains a unique geodesic representative. Therefore, one may work with these geodesic representatives instead of isotopy classes. Indeed, this is how it is done in [17]. In Chapter 4, we study the automorphisms of the complex of two-sided simple closed curves on a nonorientable surface N of genus g with n punctures such that $g + n \geq 5$. Most of the results in Chapter 4 are obtained by Atalan in the project supported by TÜBITAK - 110T665.

CHAPTER 2

PRELIMINARIES AND NOTATIONS

In this chapter, we will give basic definitions, some examples and preliminary information used in this thesis.

2.1 Some Definitions from Algebra

Definition 2.1.1 *(see pg 36, [6])*

Let $(G_1, *)$ *and* $(G_2, \circ)$ *be groups. A map from a group* G_1 *to another group* G_2 $\phi : G_1 \to G_2$ *such that*

$$\phi(x * y) = \phi(x) \circ \phi(y) \ \ for \ all \ \ x, y \in G_1$$

is said to be a homomorphism.

When the group operations for both groups are not explicitly written, the homomorphism condition can be written simply

$$\phi(xy) = \phi(x)\phi(y)$$

Definition 2.1.2 *(see pg 37, [6])*

The map $\phi : G_1 \to G_2$ *is called an isomorphism ,*G_1 *and* G_2 *are said to be isomorphic if*

(1) ϕ *is a homomorphism, and*

(2) ϕ *is a bijection.*

If G_1 is isomorphic to G_2 then we can write $G_1 \cong G_2$.

Example 2.1.3 *(see pg 37, [6])*

Let $\phi : \mathbb{R} \rightarrow \mathbb{R}^+$ be defined by $\phi(x) = e^x$, where e is the base of the natural logarithm, is an isomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}^+, .)$. Since ϕ has an inverse function (i.e., $\ln$), ϕ is a bijection. As $e^{x+y} = e^x \,.\, e^y$, the map ϕ preserves the group operations.

Definition 2.1.4 *(see pg 41, [6])*

Let G be a group and let $Aut(G)$ be the set of all isomorphisms from G onto G. $Aut(G)$ is a group under function composition. We call it the automorphism group of G and the elements of $Aut(G)$ are called automorphisms of G.

Definition 2.1.5 *(see pg 41, [6])*

A group action of a group G on a set X is a map $\varphi : G \times X \rightarrow X$ satisfying the following conditions :

(1) $g\,.\,(h\,.\,x) = (gh)\,.\,x$ *for all* $g, h \in G$, $x \in X$, *and*

(2) $1\,.\,x = x$, *for all* $x \in X$.

Note that "." is not a binary operation and $g\,.\,x$ is always an element of X.

2.2 Some Definitions and Facts from Topology

Definition 2.2.1 *(see pg 4, [11])*

Let X and Y be topological spaces. A function $\phi : X \rightarrow Y$ is called a homeomorphism if it is a bijection, continuous and the inverse function ϕ^{-1} is continuous. We say that X is homeomorphic to Y if there is a homeomorphism between X and Y.

Example 2.2.2 *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}^+$ be defined by $\phi(x) = e^x$ as in Example 2.1.3. This function is a homeomorphism because ϕ is bijection, continuous and its inverse (i.e., $\ln$) is continuous. Then $\mathbb{R}$ is homeomorphic to $\mathbb{R}^+$.*

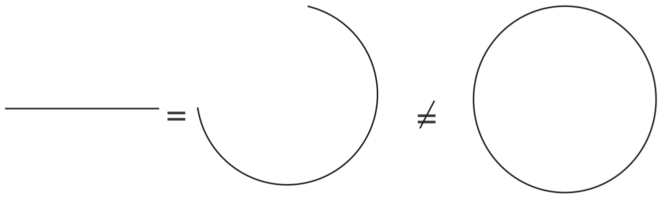

Figure 2.1: A line is homeomorphic to an arc but not to S^1.

Definition 2.2.3 *(see pg 67, [11]) A topological space is called an n-dimensional manifold if every point has a neighborhood homeomorphic to an n-dimensional open disc. We also ask that any two different points have disjoint neighborhoods.*

Definition 2.2.4 *(see pg 67, [11]) A 2-dimensional manifold is said to be a surface.*

Definition 2.2.5 *(see pg 70, [11]) A nonorientable surface is one which contains a Möbius band.*

Example 2.2.6 *The sphere and the torus are orientable surfaces.*

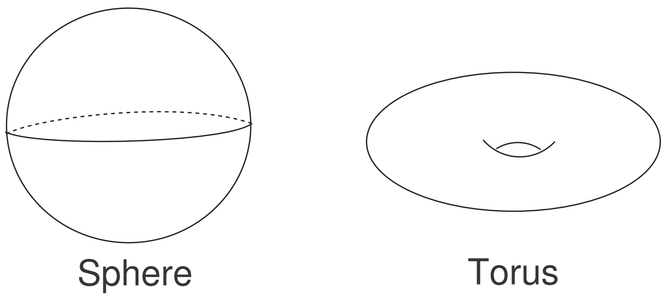

Figure 2.2: Orientable Surfaces

Example 2.2.7 *The Möbius band and the Klein bottle are nonorientable surfaces.*

Theorem 2.2.8 *(Classification of Surfaces)(see pg 79, [11])*

Every compact connected surface is homeomorphic to a sphere, a connected sum of n tori, or a connected sum of n projective planes.

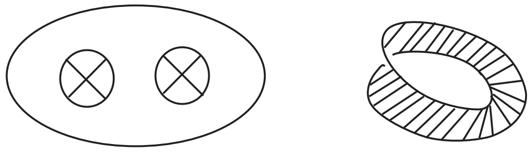

Figure 2.3: Nonorientable Surfaces

Theorem 2.2.9 *(see pg 87, [11])*

A compact connected surface with boundary is topologically equivalent to a sphere, the connected sum of n tori, or the connected sum of n projective planes, with a finite number of discs removed.

Example 2.2.10 *The cylinder and the Möbius band are surfaces with boundary.*

Definition 2.2.11 *(see pg 197, [11])*

Let X and Y be two topological spaces and f and g be two continuous functions from X to Y. Then f is homotopic to g if there is a continuous family of continuous functions $f_t : X \to Y$ for $0 \leq t \leq 1$, called a homotopy from f to g, satisfying

(1) $f_0 = f$

(2) $f_1 = g$

(3) $f_t(x)$ *is continuous both as a function of the pair* (x, t), $x \in X$, $t \in [0, 1]$.

If each f_t is an embedding, then f is called an isotopy.

Definition 2.2.12 *(see pg 31, [4])*

A simple closed curve is a connected compact 1-dimensional manifold.

Note that a simple closed curve on a surface is an embedded 1-dimensional submanifold. Similarly, by arc we consider submanifolds homeomorphic to the interval $[0, 1]$.

Definition 2.2.13 *(see pg 7, [9])*

A simple closed curve c on a surface Σ is called trivial if the curve c bounds a disc, a disc with one puncture, a Möbius band or is homotopic to some boundary component of Σ.

Similarly, we say that an arc is trivial if it can be deformed into the boundary of the surface Σ in such a way that its boundary stays in the boundary of the surface Σ.

Definition 2.2.14 *(see pg 7, [9])*

We call a simple closed curve c on a surface Σ nonseparating if $\Sigma \setminus c$ is connected, and separating otherwise.

Definition 2.2.15 *(see pg 28, [7])*

The geometric intersection number between free homotopy classes γ and δ of simple closed curves on a surface Σ is defined to be the minimal number of intersection points between a representative curve in the class γ and a representative curve in the class δ:

$$i(\gamma, \delta) = \min\{\ |c \cap d| : c \in \gamma, d \in \delta\}.$$

2.3 The Mapping Class Group of a Surface

Definition 2.3.1 *(see pg 44, [7])*

Let S be an orientable surface. Let $Homeo^{+}(S)$ denote the group of orientation-preserving homeomorphisms of S which restrict to the identity on the boundary of S.

The mapping class group of S, denoted $MCG(S)$, is the group of isotopy classes of elements of $Homeo^{+}(S)$, where isotopies are required to fix the boundary pointwise. Let $Homeo_0(S)$ be the connected component of the identity in $Homeo^{+}(S)$, then we write

$$MCG(S) = Homeo^{+}(S)/Homeo_0(S)$$

We note that elements of $MCG(S)$ are called mapping classes. If we consider orientation-reversing homeomorphisms into the definition the group we obtain is called the extended mapping class group. We denote it by $MCG^{*}(S)$.

Remark 2.3.2 *If S is a surface with punctures, then $MCG(S)$ is the group of homeomorphisms of S which leave the set of punctures invariant, modulo isotopies that leave the set of punctures invariant.*

Example 2.3.3 *(see pg 47, [7])*

Let S be a disc. The group $MCG(S)$ is trivial.

The mapping class group of a disc with one puncture is also trivial.

We note that a pair of pants is a sphere with 3 boundary components.

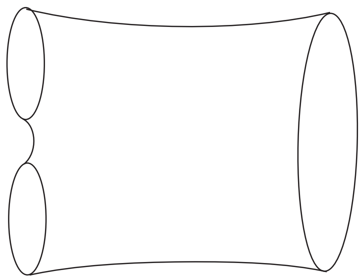

Figure 2.4: A pair of pants

Example 2.3.4 *(see pg 50, [7])*

Let S be a pair of pants. The group $MCG(S)$ is isomorphic to $\mathbb{Z}^3$.

Example 2.3.5 *(see pg 53, [7])*

Let S be the torus. The group $MCG(S)$ is isomorphic to the classical modular group $SL(2, \mathbb{Z})$ of 2×2 integral matrices with determinant 1.

Definition 2.3.6 *(see pg 4, [12]) Let A be the cylinder* $[0,1] \times S^1$ *with coordinates t and* $e^{i\theta}$*. Let us define a map* $\varphi : A \to A$ *by*

$$\varphi(t, e^{i\theta}) = (t, e^{i(\theta+2t\pi)}).$$

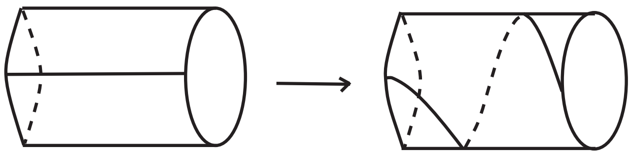

Figure 2.5: The twist φ on A

The map $\varphi : A \to A$ *is a homeomorphism and the restriction* $\varphi_{|\partial A}$ *is the identity, where* ∂A *is the boundary of A.*

Let c be a simple closed curve on S such that a regular neighborhood of c is an annulus. Let U be a closed neighborhood of c homeomorphic to A. Let us pick an oriented-preserving embedding $\rho : A \to S$ *such that* $\rho(A) = U$*. Let us define a homeomorphism* $T_c : S \to S$ *by*

$$T_c(x) = \begin{cases} \rho\varphi\rho^{-1}(x) & \text{if } x \text{ is in } \rho(A) \\ x & \text{if } x \text{ is not in } \rho(A). \end{cases}$$

We can think the effect of T_c *as cutting the surface S along the simple closed curve c, to twist one of the side by* 2π *to the right on the positive side of the surface S and to reglue it.*

We note that if the simple closed curve c is isotopic to another simple closed curve d, then T_c *and* T_d *are isotopic.*

We denote by t_c *the isotopy class of the homeomorphism* T_c*. The isotopy class* t_c *is called the positive Dehn twist about the simple closed curve c.*

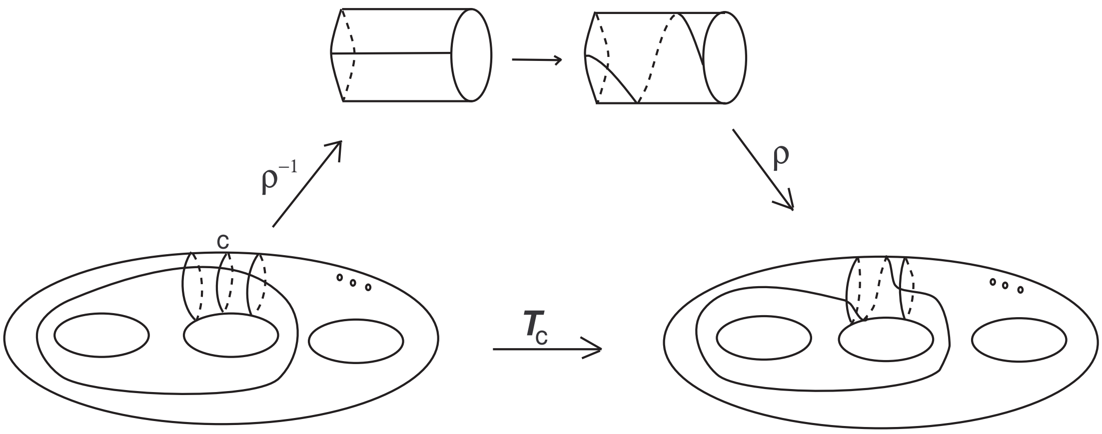

Figure 2.6: The Dehn twist T_c on the surface

2.4 Graphs and Complexes of Curves

Definition 2.4.1 *(see pg 99, [4])*

A graph consists of a set V, whose elements are called the vertices of the graph, and a set E of some two element subsets of V. The elements of E are said to be the edges of the graph.

Definition 2.4.2 *(see pg 100, [4])*

A device for showing a graph is called a graph diagram. A graph diagram for a graph shows dots in the plane to represent the graph's vertices and lines or arcs in the plane that connect the dots to represent the graph's edge.

Definition 2.4.3 *(see pg 22, [9])*

A simplicial complex constitutes a set of vertices and a set of simplices. Simplices are nonempty finite sets of vertices such that a nonempty subset of a simplex is a simplex; every vertex belongs to some simplex. The dimension of a simplex is the number of vertices minus 1.

Definition 2.4.4 *(see pg 22, [9])*

Curve complexes are abstract simplicial complexes. The vertices of the complex of curves of a surface are the isotopy classes of nontrivial simple closed curves on the surface. Any two vertices corresponding to isotopy classes with the geometric intersection number zero are connected by an edge. A set of vertices $\{u_1, \ldots, u_{n+1}\}$ *is said to be a* $n-$*simplex if and only if they are disjoint pairwisely.*

CHAPTER 3

AUTOMORPHISM GROUP OF THE GRAPH $G_{ph}(S)$ ON ORIENTABLE SURFACES

We consider an orientable surface S of genus $g \geq 1$ with n punctures. In this chapter, we study Schaller's work related to the automorphisms of a certain graph.

3.1 Introduction

Definition 3.1.1 *(i) A (g, n)-surface is an orientable surface of genus g with n punctures.*

(ii) A boundary component of a surface is a simple closed curve or a puncture.

(iii) Let S be a (g, n)-surface. An embedded subsurface $S_1 \subset S$ is said to be a (h, k)-subsurface, where S_1 has genus h with k boundary components.

Remark 3.1.2 *Let S be a (g, n)-surface and c be a nonseparating simple closed curve of S. We use the surface that is the closure of $S_1 = S \setminus c$. We do not distinguish S_1 and its closure.*

Definition 3.1.3 *Let S be a (g, n)-surface, $g \geq 1$ and $n \geq 0$.*

(1) Let $V(S)$ denote the set of nonseparating simple closed curves of the surface S.

(2) Let α and β be any two nonseparating simple closed curves. We denote the number of geometric intersection points of α and β by $\mathrm{i}(\alpha, \beta)$. If α is equal to β,

then $i(\alpha,\beta) = 0$. On the other hand, if α is not equal to β and $i(\alpha,\beta) = 0$, then we say that α and β are disjoint.

(3) *We say that α and β are dual, if $i(\alpha,\beta) = 1$. We shall write $\alpha \perp \beta$.*

Definition 3.1.4 *Let S be a (g,n)-surface.*

(1) *Let $G_{ph}(S)$ denote the following graph:*
$V(S)$ is the set of vertices of $G_{ph}(S)$ and

$$\{(\alpha,\beta) \in V(S) \times V(S) : \alpha \perp \beta\}$$

is the set of (nonoriented) edges.

(2) *Let $Y = \{\alpha_1, \alpha_2, \ldots, \alpha_r\} \subset V(S)$, $r \geq 1$. Let us define*

$$N(Y) := N(\alpha_1, \alpha_2, \ldots, \alpha_r) := \{x \in V(S) : x \perp \alpha_i,\ for\ all\ i = 1, 2, \ldots.., r\}.$$

(3) *We denote the automorphism group of the graph $G_{ph}(S)$ by $Aut(G_{ph}(S))$.*

Definition 3.1.5 *Let S be a (g,n)-surface, $g \geq 1$. Let α and β be in $V(S)$ such that $i(\alpha,\beta) \geq 2$. Let β_1 be a connected component of β in $S \setminus \alpha$. Suppose that $S \setminus (\alpha \cup \beta_1)$ is connected. Then, we say that β_1 is a nonseparating component of β with respect to α. Otherwise, β_1 is said to be a separating component of β with respect to α.*

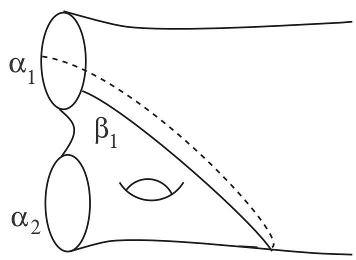

Figure 3.1: β_1 is a separating component of β with respect to α

Definition 3.1.6 *Let S be a (g,n)-surface, $g \geq 1$. Let $\alpha, \beta \in V(S)$ such that $i(\alpha,\beta) \geq 2$. Let S_1 denote $S \setminus \alpha$. Let β_1 be a nonseparating component of β with respect to α. Let α_1 and α_2 denote the two copies of α in S_1. If β_1 relates α_1 and α_2, then we say that β_1 is two-sided arc. Otherwise, β_1 is said to be one-sided arc.*

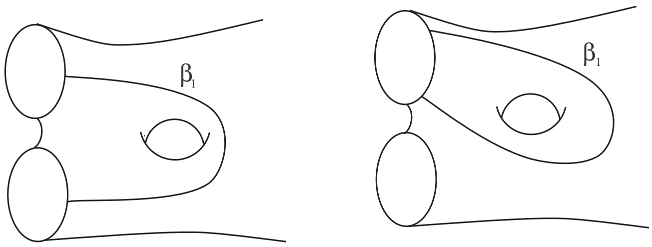

Figure 3.2: β_1 is two-sided arc on the left-hand side and one-sided arc on the right-hand side

3.2 $G_{ph}(S)$ recognizes whether the elements are disjoint or not disjoint

Lemma 3.2.1 *Let S be a (g,n)-surface, $g \geq 1$. Let $\alpha, \beta \in V(S)$ such that $i(\alpha,\beta) \geq 2$. Let β_1 be a separating component of β with respect to α. Then there exists $\gamma \in V(S) \setminus \{\alpha,\beta\}$ such that $N(\alpha, \beta) \subset N(\gamma)$. Furthermore, $i(\alpha,\gamma) = 0$.*

Proof. Let $S_1 = S \setminus \alpha$. Let α_1, α_2 be the two copies of α in S_1. Then β_1 begins and finishes in the same boundary component of S_1, in α_1 as in Figure 3.3. Let B_1 and B_2 be two connected components of $S_1 \setminus \beta_1$ such that α_2 lies in B_2. There is a unique simple closed curve γ in B_2 such that $B_2 \setminus \gamma$ has a connected component C of genus $g = 0$ with no punctures and which has β_1 in its boundary. Since β is a simple curve, γ is not equal to α_2. Thus, $\gamma \in V(S) \setminus \{\alpha,\beta\}$. Let us define $U = C \cup B_1$.

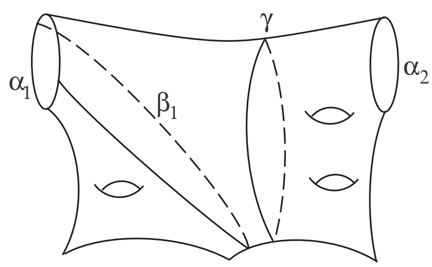

Figure 3.3: The separating component β_1 with respect to α in S_1

Let $a \in N(\alpha,\beta)$. Since $i(a,\alpha) = 1$, thus, $a \cap U$ has a connected component a_1 relating α_1 and γ. Suppose that $a \cap U$ has another connected component a_2. Then a_2 begins and finishes in γ and so, intersects β_1 at least twice since β_1 separates U. However, since $i(a,\beta) = 1$, we get a contradiction. Hence, a_2 cannot exist, and so, $i(a,\gamma) = 1$. In other words, $a \in N(\gamma)$. We obtain that $N(\alpha,\beta) \subset N(\gamma)$. □

Definition 3.2.2 *Let α, β and γ be three elements of $V(S)$. A subset $\{\alpha, \beta, \gamma\}$ is said to be a triple if*

(1) the three elements are mutually dual,

(2) S has a $(1,1)$-subsurface which contains α, β and γ.

Fact 3.2.3 *Let S be a (g,n)-surface, $g \geq 1$. Let α and β be in $V(S)$ such that $i(\alpha,\beta) = 1$. Then $V(S)$ has exactly two distinct elements γ such that $\{\alpha, \beta, \gamma\}$ is a triple. Furthermore, $S \setminus (\alpha \cup \beta \cup \gamma)$ has 3-connected components, two of them are isometric hyperbolic triangles.*

Lemma 3.2.4 *Let S be a (g,n)-surface, $g \geq 1$. Let α and β be in $V(S)$ such that $i(\alpha,\beta) \geq 2$. Suppose that β_1 be a nonseparating component of β with respect to α. Then there are γ, γ' in $V(S) \setminus \{\alpha,\beta\}$ such that $N(\alpha,\beta) \subset (N(\gamma) \cup N(\gamma'))$. Furthermore, suppose that β_1 is one-sided arc. Then α, γ and γ' are pairwise disjoint. On the other hand, if β_1 is two-sided arc, then $\{\alpha, \gamma, \gamma'\}$ is a triple with*

$$i(\alpha,\beta) = i(\beta,\gamma) + i(\beta,\gamma') \text{ and } \min\{i(\beta,\gamma), i(\beta,\gamma')\} > 0 \tag{3.1}$$

Proof.

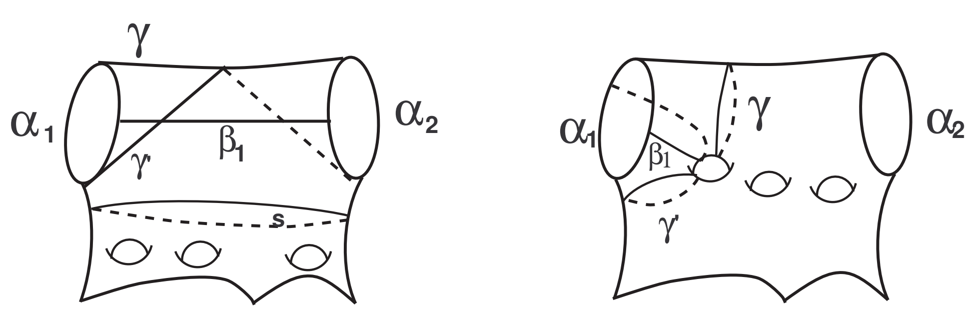

Figure 3.4: The nonseparating component β_1 of β with respect to α in S_1

Let α_1 and α_2 be the two copies of α in S_1. (see Figure 3.4)

(1) Suppose that β_1 is two-sided arc. Let us cut S_1 along β_1. Then in the boundary of the obtained surface there exists a simple closed curve freely homotopic to a unique simple closed curve s (in S_1) that is the boundary curve of a pair of pants W (in S_1) containing β_1. α_1 and α_2 are two other boundary curves of W. In S, s separates a $(1,1)$-subsurface M from the rest. We note that M consists of α.

Let a be in $N(\alpha,\beta)$. Let a_1 be the connected component of $a \cap M$ intersecting α. Suppose that $a \cap M$ has another connected component a_2. Since $a \in N(\alpha)$, a_2 does not cross α. By construction, a_2 crosses β_1. Also, because $a \in N(\beta)$, $a \cap M$ has at most two connected components.

Now, let γ and γ' be simple closed curves in M such that $\{\alpha, \gamma, \gamma'\}$ is a triple, $\gamma \cap W$ and $\gamma' \cap W$ are homotopic to β_1. It follows that β_1 crosses every of γ and γ' at most one point. Let a_1 be disjoint from β_1. Then, by Fact 3.2.3, a_1 is disjoint from one of γ and γ'. And also it crosses one point the other one. If there is no a_2, we are done. If a_2 exists, then it crosses one point every of γ, γ' and it is done. So suppose that a_1 crosses β_1 (so, a_2 does not exist). It follows from Fact 3.2.3 that a_1 crosses once one of γ and γ'. Hence, we have proved that $a \in N(\gamma) \cup N(\gamma')$.

By Fact 3.2.3, the triangle inequality and the fact that β cannot cross transversally β_1. So, we get $i(\alpha,\beta) = i(\beta,\gamma)+i(\beta,\gamma')$. Let β' be the connected component of β in M containing β_1. Then β' crosses α at least two points. Therefore, β' cannot be connected in $M \setminus \gamma$ nor in $M \setminus \gamma'$. This shows that β crosses both γ and γ'. Hence Equation 3.1 holds.

(2) Now suppose that β_1 begins and finishes in α_1. Then α_1 is separated by β_1 into two parts α_{1a} and α_{1b}. Let γ be the simple closed curve in S_1 freely homotopic to $\alpha_{1a} \cup \beta_1$. Let γ' be the simple closed curve in S_1 freely homotopic to $\alpha_{1b} \cup \beta_1$. Then α_1, γ and γ' are the boundary curves of a unique pair of pants W, embedded in S_1. We note that W contains β_1. Since β_1 is a nonseparating component, γ and γ' are in $V(S) \setminus \{\alpha,\beta\}$. Let $a \in N(\alpha,\beta)$. We obtain that $a \cap W$ accepts a connected component a_1 begining in α_1 and finishing in γ or in γ'. Let a_2 denote a second connected component of $a \cap W$. Then a_2 must relate γ and γ'. Therefore it crosses β_1. It implies that $a \cap W$ has at most two connected

components. Thus, $a \in N(\gamma) \cup N(\gamma')$.

$\square$

Proposition 3.2.5 *Let S be a (g, n)-surface, $g \geq 1$. Then the graph $G_{ph}(S)$ is connected.*

Proof. Let α and β be two elements of $V(S)$. Let $r = i(\alpha, \beta)$. If $r = 0$, then $N(\alpha, \beta) \neq \emptyset$. If $r > 1$, then there exists γ which is defined as in Lemma 3.2.1 or in Lemma 3.2.4 such that $i(\beta, \gamma) < r$. By induction on r, β and γ are in the same connected component of the graph $G_{ph}(S)$. Hence, also α and β are in the same connected component of the graph $G_{ph}(S)$. Thus, we obtain that the graph $G_{ph}(S)$ is connected. $\square$

Lemma 3.2.6 *Let S be a (g, n)-surface. Let $Y \subset V(S)$. Suppose that there is $\beta \in V(S)$ such that α and β are disjoint for all $\alpha \in Y$. If $N(Y)$ has an element γ intersecting β, then $N(Y)$ is infinite.*

Proof. Let $\gamma \in N(Y)$ such that γ crosses β. Let us apply a full twist deformation along β. The result is a surface S_1 isometric to S. This twist deformation does not change the elements of Y; however, γ has become a distinct element $\gamma_1 \in V(S)$. Certainly, $\gamma_1 \in N(Y)$. The similar argument holds for a twist deformation along β of r full twists ($r \in \mathbb{Z}$). The proof of the lemma is finished. $\square$

Lemma 3.2.7 *Let S be a (g, n)-surface, $g \geq 1$. Let $\alpha, \beta \in V(S)$ with $i(\alpha, \beta) = 0$. Then there are no elements $\gamma, \gamma' \in V(S) \setminus \{\alpha, \beta\}$ such that $N(\alpha, \beta) \subset (N(\gamma) \cup N(\gamma'))$.*

Proof. Suppose that there are $\gamma, \gamma' \in V(S) \setminus \{\alpha, \beta\}$ such that $N(\alpha, \beta) \subset (N(\gamma) \cup N(\gamma'))$. Suppose that $s \in V(S)$ such that $i(\gamma, s)$ is positive, the geometric intersection numbers $i(\alpha, s)$ and $i(\beta, s)$ are equal to zero. Similarly, suppose that another element $s' \in V(S)$ such that $i(\gamma', s')$ is positive, $i(\alpha, s')$ and $i(\beta, s')$ are equal to zero. Let $T \subset N(\alpha, \beta)$ be the subset of elements intersecting both s and s'. It is obvious that $T \neq \emptyset$.

Let t be an element of T. By Lemma 3.2.6 we may twist t along s to get $t' \in T$ such that $i(t', \gamma)$ becomes big. By the similar argument, we also may twist t' along s' so that $i(t', \gamma')$ becomes big. Thus, this is impossible that $T \subset (N(\gamma) \cup N(\gamma'))$. $\square$

Now, it follows from Lemma 3.2.1, Lemma 3.2.4 and Lemma 3.2.7 that we have the following result :

Theorem 3.2.8 *Let S be a (g,n)-surface, $g \geq 1$. Let $\alpha, \beta \in V(S)$, such that α is not equal to β. Then, for each $h \in Aut(G_{ph}(S))$*

$$i(\alpha,\beta) = 0 \Leftrightarrow i(h(\alpha), h(\beta)) = 0.$$

In other words, $G_{ph}(S)$ recognizes whether the elements of nonseparating simple closed curves are disjoint or not disjoint.

3.3 An automorphism preserves the geometric intersection numbers

Definition 3.3.1 *Let S be a (g,n)-surface, $g \geq 1$. If $P \subset V(S)$ is a set of $3g-3+n$ pairwise disjoint elements, then P is said to be a partition of S.*

Convention 3.3.2 *Let S be a (g,n)-surface. Let $f \in MCG^*(S)$, taken as a homeomorphism of S. If α is a simple closed curve of S, then $f(\alpha)$ is also a simple closed curve in S, and in the homotopy class of $f(\alpha)$, there is a uniqe simple closed geodesic. So, f induces a map, denoted by f. We note that if f is replaced by f' isotopic to f, this map does not change. We use this interpretation of the elements of $MCG^*(S)$.*

Corollary 3.3.3 *Let S be a (g,n)-surface, $g \geq 1$. Let P be a partition. Let h be an element of $Aut(G_{ph}(S))$. Then $h(P)$ is also a partition. Furthermore, there is a mapping class f in $MCG^*(S)$ such that $f(\alpha) = h(\alpha)$ for all $\alpha \in P$.*

Proof. It follows from Theorem 3.2.8 that $h(P)$ is a partition. So, it is sufficient to show that h maps the boundary components of a pair of pants (induced by P), to the boundary components of a pair of pants (induced by $h(P)$). This implies the existence of f as claimed.

Let Q be a pair of pants induced by P with boundary components α, β and γ.

(1) Suppose that α, β and γ are in P. Suppose also that $N(\alpha,\beta,\gamma)$ is empty, so $N(h(\alpha), h(\beta), h(\gamma))$ is also empty. It follows that $S \setminus (h(\alpha), h(\beta), h(\gamma))$ is dis-

connected. Further, there is $\alpha' \in N(\alpha)$ such that $i(\alpha', \beta) = 0$. Then, by Theorem 3.2.8, $i(h(\alpha'), h(\beta)) = 0$. It follows that $S \setminus (h(\alpha) \cup h(\beta))$ is connected. By the similar argument, we have $S \setminus (h(\alpha) \cup h(\gamma))$ and $S \setminus (h(\beta) \cup h(\gamma))$ are connected. Thus, $h(\alpha)$, $h(\beta)$ and $h(\gamma)$ are the boundary components of a pair of pants.

(2) Now suppose that γ is a puncture and that only $\alpha, \beta \in P$. Then there is no α' in $N(\alpha)$ with $i(\alpha', \beta) = 0$. By Theorem 3.2.8, h satisfies this property. This implies that $S \setminus (h(\alpha) \cup h(\beta))$ is not connected.

On the other hand, there is s in $V(S)$ disjoint from $\alpha \cup \beta$ such that s crosses all elements of $P \setminus \{\alpha, \beta\}$. By Theorem 3.2.8, h satisfies this property. Thus, the boundary components of a pair of pants induced by $h(P)$ are $h(\alpha)$ and $h(\beta)$.

□

Corollary 3.3.4 *Let S be a (g, n)-surface, $g \geq 1$. Let $\{\alpha, \beta, \gamma\} \subset V(S)$ be a triple. Then $\{h(\alpha), h(\beta), h(\gamma)\}$ is also a triple for each $h \in Aut(G_{ph}(S))$.*

Proof. By definition of a triple, there is a $(1, 1)$-subsurface M of S with boundary component s such that α, β and γ are in M. We can exclude $(g, n) = (1, 1)$, because the corollary holds this case. And also, we may suppose that s is a simple closed curve.

(1) Let $g > 1$. Then there is a $(1, 2)$-subsurface L of S with boundary components $u, v \in V(S)$ containing M. Furthermore, there is t in $N(u, v)$ with $i(t, s) = 0$. By Corollary 3.3.3, $h(\alpha)$, $h(\beta)$ and $h(\gamma)$ are in a $(1, 2)$-subsurface L' of S with boundary components $h(u)$ and $h(v)$. Since $i(h(t), h(\alpha)) = i(h(t), h(\beta)) = i(h(t), h(\gamma)) = 0$ and $h(t)$ also crosses the boundary of L'. Thus, $h(\alpha)$, $h(\beta)$ and $h(\gamma)$ are in a $(1, 1)$-subsurface.

(2) Now suppose that $g = 1$ with n punctures. Then S has a partition

$$P = \{\alpha, u_1, \ldots, u_{n-1}\} \subset V(S)$$

such that $u_i \in N(\beta, \gamma)$, for every $i = 1, \ldots, n-1$. To u_i there is a unique simple closed curve $s_i \in S$ such that $i(s_i, \beta) = 0$ and it is disjoint from all elements of $P \setminus \{u_i\}$, $i = 1, \ldots, n-1$. Because $\{\alpha, \beta, \gamma\}$ is a triple, $i(\gamma, s_i) = 0$, for $i = 1, \ldots, n-1$. It follows from Lemma 3.2.6 that if the elements of $P \setminus \{u_i\}$ are fixed, there exist infinitely many distinct possibilities to pick u_i with the desired properties. By Corollary 3.3.3, $P' = h(P)$ is a partition. To pick u_i, because there are infinitely many distinct possibilities we obtain that there is a simple closed curve s' disjoint from $h(\beta)$, $h(\gamma)$ and from all elements of $P' \setminus \{u'\}$ for each $u' \in P' \setminus \{h(\alpha)\}$. It follows that $\{h(\alpha), h(\beta), h(\gamma)\}$ is a triple.

□

Theorem 3.3.5 *Let S be a (g, n)-surface, $g \geq 1$. Let h be an element of $Aut(G_{ph}(S))$. Let α and β be any two elements of $V(S)$. Then $i(\alpha, \beta)$ is equal to $i(h(\alpha), h(\beta))$.*

Proof. Let $i(\alpha, \beta)$ be equal to zero. The theorem follows by Theorem 3.2.8. Suppose that the theorem holds for all $\alpha, \beta \in V(S)$ with $i(\alpha, \beta) \leq r - 1$, for a $r \geq 2$.

Let α, $\beta \in V(S)$ such that $i(\alpha, \beta) = r$. We need to prove that $i(\alpha, \beta)$ is equal to $i(h(\alpha), h(\beta))$. We think of a component of β as a component with respect to α.

(i) Suppose that there is a two-sided arc component β_1 of β. Let γ and γ' be defined as in Lemma 3.2.4. Then $\{\alpha, \gamma, \gamma'\}$ is a triple and Equation 3.1 in Lemma 3.2.4 holds. Since β crosses both γ and γ', we obtain that $i(\beta, \gamma) = i(h(\beta), h(\gamma))$ and $i(\beta, \gamma') = i(h(\beta), h(\gamma'))$ by induction hypothesis. Since $\{\alpha, \gamma, \gamma'\}$ is a triple, by Corollary 3.3.4, then $\{h(\alpha), h(\gamma), h(\gamma')\}$ is a triple. Thus, by the triangle inequality and Fact 3.2.3,

$$i(h(\beta), h(\gamma)) + i(h(\beta), h(\gamma')) \geq i(h(\alpha), h(\beta))$$

which gives $i(\alpha, \beta) \geq i(h(\alpha), h(\beta))$. Thus, by induction hypothesis on r (applied to h^{-1}), we have $i(\alpha, \beta) = i(h(\alpha), h(\beta))$.

(ii) Let α_1 and α_2 be the two copies of α in $S' = S \setminus \alpha$. By (i), we may suppose that all components of β are separating or one-sided arc. Let S_i denote the smallest embedded subsurface of S' (the boundary components of S_i being simple closed curves or punctures) such that all components of β with endpoints on α_i are contained in S_i, for $i = 1,2$. Then the interiors of S_1 and S_2 are disjoint. Because of nonseparating α, S_1 and S_2 have a common boundary component $u = u_1 = u_2 \in V(S)$ or $S' \setminus (S_1 \cup S_2)$ has a connected component S_3 having a common boundary component $u_i \in V(S)$ with S_i, for $i = 1,2$. Then there is a simple curve ω_i in S_i relating α_i and u_i and is disjoint from β, for $i = 1,2$. Let ω be a simple curve in S' relating α_1 and α_2 such that $\omega \cap S_i = \omega_i$, for $i = 1,2$. Let t be a curve segment homotopic to ω. We consider t as a component of a simple closed curve with respect to α. Then, we define γ and γ' as in Lemma 3.2.4. As in Lemma 3.2.4, we obtain that $i(\alpha,\beta) = i(\beta,\gamma) + i(\beta,\gamma')$. If β crosses both γ and γ', by the similar argument as in (i), we have $i(\alpha,\beta) = i(h(\alpha), h(\beta))$.

Suppose that $g = 1$. β must cross both γ and γ'. Because otherwise, β is separating. This gives a proof of the theorem for $g = 1$.

Suppose that $g > 1$. We note that the role of α and β can be interchanged and by the similar argument as above, we can construct a triple $\{\alpha, \overline{\gamma}, \overline{\gamma'}\}$ such that $i(\alpha,\beta) = i(\alpha,\overline{\gamma}) + i(\alpha,\overline{\gamma'})$, where $\overline{\gamma}$ and $\overline{\gamma'}$ are dual to u_1. Therefore, we may suppose that β is disjoint to γ and that α is disjoint to $\overline{\gamma}$. This gives that α and β are nonseparating in $\Omega = S \setminus u_1$. Ω is homeomorphic to a $(g-1, n+2)$-surface, which is also denoted by Ω. Certainly, an element in $Aut(G_{ph}(\Omega))$ is induced by h canonically. By induction on the genus g, we obtain that $i(\alpha,\beta) = i(h(\alpha), h(\beta))$.

□

Fact 3.3.6 *Let S be a $(0,4)$-surface. Let α and β be simple closed curves of S such that $i(\alpha,\beta) = 2$.*

(i) Let γ be a simple closed curve of S with $i(\alpha,\gamma) = 2$. Then there is $f \in MCG^*(S)$ such that $f(\alpha) = \alpha$ and $f(\beta) = \gamma$.

(ii) There are exactly two simple closed curves γ_i of S with $i(\alpha,\gamma_i) = 2$ and $i(\beta,\gamma_i) = 2$, for $i = 1,2$. Furthermore, there is $f \in MCG^*(S)$ such that $f(\alpha) = \alpha$, $f(\beta) = \beta$ and $f(\gamma_1) = \gamma_2$.

Theorem 3.3.7 *Let S be a (g,n)-surface, $g \geq 1$. Suppose that $(g,n) \notin \{(1,1),(1,2),(2,0)\}$. Then*

$$Aut(G_{ph}(S)) \cong MCG^*(S).$$

Suppose that $(g,n) \in \{(1,1),(1,2),(2,0)\}$. Then

$$Aut(G_{ph}(S)) \cong MCG^*(S)/H,$$

where H is the subgroup generated by the hyperelliptic involution.

Proof. Let $f \in MCG^*(S)$. By Convention 3.3.2, $f \in Aut(G_{ph}(S))$. Thus, there is a group homomorphism

$$\Gamma(g,n) : MCG^*(S) \to Aut(G_{ph}(S)).$$

(1) In these three cases $(g,n) \in \{(1,1),(1,2),(2,0)\}$ the kernel of $\Gamma(g,n)$ is nontrivial. Because the isotopy class of the identity and the isotopy class of the unique hyperelliptic involution is contained in the kernel of $\Gamma(g,n)$.

Otherwise, the kernel of $\Gamma(g,n)$ is trivial.

(2) We must show that $\Gamma(g,n)$ is onto. Let $h \in Aut(G_{ph}(S))$. Let $k = 3g-3+n$. Let $P = \{\alpha_1,\ldots,\alpha_k\} \subset V(S)$ be a partition of the surface S. By Corollary 3.3.3, we may suppose that $h(\alpha_i) = \alpha_i,\ i = 1,\ldots,k$. To each α_i, $i = 1,\ldots,k$, there is $\beta_i \in V(S)$ such that $i(\alpha_i,\beta_i)$ is equal to 2 and $i(\alpha_j,\beta_i)$ is equal to zero for all $j \neq i$, $j = 1,\ldots,k$. By Fact 3.3.6 (i) we may suppose that $h(\beta_i) = \beta_i,\ i = 1,\ldots,k$.

For $i = 1,\ldots,k$, there is $\gamma_i \in V(S)$ with $i(\alpha_j,\gamma_i) = 0$ for all $j \neq i,\ j = 1,\ldots,k$ and with $i(\alpha_i,\gamma_i) = i(\beta_i,\gamma_i) = 2$. By Fact 3.3.6 ($ii$) we may suppose that $h(\gamma_i) = \gamma_i,\ i = 1,\ldots,k$.

Let $P' = \{\alpha_i,\beta_i,\gamma_i :\ i = 1,\ldots,k\}$. Now since $u \in V(S)$ is uniquely determined by the $3k$ intersection numbers $i(u,v)$, $v \in P'$ by Dehn's work and Thurston's work (see [5], [16]). Consequently, $h(u) = u$ for every $u \in V(S)$ by Theorem 3.3.5. Hence, h is the identity and $\Gamma(g,n)$ is onto.

□

CHAPTER 4

AUTOMORPHISMS OF COMPLEXES OF TWO-SIDED CURVES ON NONORIENTABLE SURFACES

In this chapter, we study automorphisms of two-sided curve complexes of a nonorientable surface N of genus g with n punctures. In this chapter, we denote two-sided simple closed curves and their isotopy classes by the same letter.

4.1 Introduction

Let N be a compact, connected nonorientable surface of genus g with n punctures. The mapping class group $MCG(N)$ of a nonorientable surface N is defined as the group of isotopy classes of all diffeomorphisms $N \to N$.

Let a be a simple closed curve on a nonorientable surface N. If a regular neighborhood of a is an annulus or a Möbius band we say that the curve a is a two-sided simple closed curve or one-sided simple closed curve, respectively.

Let $C_T(N)$ denote the complexes of two-sided curves on a nonorientable surface N. We note that this complex is a subcomplex of $C(N)$ (See Chapter 2). The group $MCG(N)$ acts on the curve complex $C_T(N)$ on N as simplicial automorphisms. We have the following group homomorphism:

$$MCG(N) \to Aut(C_T(N)).$$

The automorphism group of the curve complex of an orientable surface is isomorphic to the extended mapping class group. For a closed surface of genus $g \geq 2$, this fact is proved by Ivanov [10]. For genus $g = 2$, there is the kernel which is $\mathbb{Z}_2$ generated by the hyperelliptic involution. Korkmaz [13] showed it for lower genus cases. Also, Luo [15] proved it using different methods in all cases independently. Atalan - Korkmaz [1] showed it for nonorientable surface of genus $g + n \geq 5$. More precisely, the kernel denoted by $Ker(\Sigma)$ for a surface Σ, is almost always trivial: If Σ is the one-holed torus, the two-holed torus, or the closed surface of genus two, then $Ker(\Sigma)$ is isomorphic to $\mathbb{Z}_2$ and generated by the hyperelliptic involution. If Σ is the four-holed sphere, then $Ker(\Sigma)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and generated by two hyperelliptic involutions.

Definition 4.1.1 *If a simple closed curve a is separating and if it bounds a disc with two punctures, then it is called a 2-separating curve.*

Definition 4.1.2 *A collection of curves $C = \{c_1, c_2, \ldots, c_k\}$ is called a multicurve if $c_1, c_2, \ldots, c_k$ are isotopy classes of nontrivial, pairwise nonisotopic and disjoint two-sided simple closed curves and none is isotopic to a boundary component of N.*

Definition 4.1.3 *A multicurve C is called a $P-S$ decomposition if every $c \in C$ is two-sided and every component of $N \setminus C$ is diffeomorphic to one of the following surfaces: a disc with two punctures, an annulus with one puncture, a sphere with three holes, a Möbius band with one puncture, or a Möbius band with one boundary component.*

Remark 4.1.4 *A $P-S$ decomposition corresponds to a maximal simplex in $C_T(N)$.*

Definition 4.1.5 *Let a and b be two distinct isotopy classes of two-sided simple closed curves in a $P-S$ decomposition on N. We say that they are adjacent to in the $P-S$ decomposition if they are on the boundary of one of the surfaces in Definition 4.1.3 complementary to $P-S$.*

Definition 4.1.6 *Let a and b be two distinct isotopy classes of two-sided simple closed curves on N. Let U denote a regular neighborhood of $a \cup b$ in N. We reglue all the discs complementary to U and all the annuli with one boundary component contained in the boundary of N and the other in U. The subsurface obtained this way is well defined up to homotopy equivalence and we say that this subsurface is filled by a and b.*

Note that if c is another isotopy class of two-sided simple closed curve that enters this subsurface, it must cross either a or b.

Definition 4.1.7 *If two isotopy classes of two-sided simple closed curves together fill either a four holed sphere or a one holed torus, we say that they have small intersection.*

Definition 4.1.8 *The complexity of N is defined as the maximal number of distinct, nonisotopic, disjoint isotopy classes of two-sided simple closed curves.*

Let $\kappa(N)$ denote the complexity of N. We note that for $g + n > 2$, if g is odd, $\kappa(N) = \frac{3}{2}(g-1) + n - 2$; if g is even $\kappa(N) = \frac{3}{2}g + n - 3$ (see [2], [3], [14]).

4.2 Automorphisms between curve complexes

Lemma 4.2.1 *Suppose that $\kappa(N) \geq 1$. Let ϕ be an automorphism of $C_T(N)$. Then, ϕ sends $P - S$ decompositions to $P - S$ decompositions.*

Proof. This follows for complexity reasons and because ϕ is a simplicial and automorphism of $C_T(N)$. □

Lemma 4.2.2 *Suppose that $\kappa(N) \geq 1$. Let ϕ be an automorphism of $C_T(N)$. Let a and b be two distinct isotopy classes of two-sided simple closed curves in N which fill either a four holed sphere or a one holed torus. Then $\phi(a)$ and $\phi(b)$ fill either a four holed sphere or a one holed torus in N.*

Proof. Let ℓ be any maximal multicurve in N such that every curve is disjoint from both a and b. For complexity reasons, $\phi(\ell)$ is a maximal multicurve disjoint from both $\phi(a)$ and $\phi(b)$. In particular, since ϕ is simplicial and an automorphism, $\phi(a)$ and $\phi(b)$ must together fill either a one holed torus or a four holed sphere. □

Lemma 4.2.3 *Suppose that $\kappa(N) \geq 1$. Let ϕ be an automorphism of $C_T(N)$ and let $\boldsymbol{P}$ be a $P - S$ decomposition on N. Then ϕ takes any two vertices in $\boldsymbol{P}$ which are not adjacent in $\boldsymbol{P}$ to two vertices which are not adjacent in $\phi(\boldsymbol{P})$.*

Proof. Let a and b be two vertices in **P** such that they are not adjacent in **P**. We can find two disjoint isotopy classes of two-sided simple closed curves c_1 and c_2 such that c_1 has small intersection with a but is disjoint from b and c_2 has small intersection with b but is disjoint from a. Assume that $\phi(a)$ and $\phi(b)$ are adjacent in $\phi(\mathbf{P})$. Then $\phi(c_1)$ and $\phi(c_2)$ must intersect, since ϕ is simplicial, we get a contradiction. □

Lemma 4.2.4 *Suppose that $\kappa(N) \geq 1$. Let ϕ be an automorphism of $C_T(N)$ and let **P** be a $P-S$ decomposition on N. Then ϕ takes any two vertices in **P** which are adjacent in **P** to two vertices which are adjacent in $\phi(\mathbf{P})$.*

Proof. Let a and b be two vertices in **P** such that a is adjacent to b in **P**. Then there is an isotopy class of two-sided curve c having small intersection with both and disjoint from each other curve in **P**. ϕ preserves this property and thus, $\phi(a)$ and $\phi(b)$ are adjacent in $\phi(\mathbf{P})$. □

Lemma 4.2.5 *Suppose that $\kappa(N) \geq 1$. Let ϕ be an automorphism of $C_T(N)$. Let ℓ be any multicurve of N. Then ϕ induces an isomorphism between the adjacency graphs of ℓ and $\phi(\ell)$.*

Proof. We use Lemma 4.2.3 and Lemma 4.2.4. We extend ℓ to a $P-S$ decomposition **P**. If two vertices are adjacent in ℓ then they either border one of the surfaces in Definition 4.1.3 with a third curve from ℓ or they border one of the surfaces in Definition 4.1.3 meeting the boundary of N. So this stays in **P**. And also, this is preserved under ϕ. We consider any two vertices not adjacent in ℓ and organize for them to be nonadjacent in **P**. So, ϕ preserves this. □

4.3 Topological types of vertices in $C_T(N)$

In this section, we distinguish vertices in the complex $C_T(N)$.

Lemma 4.3.1 *Suppose that $\kappa(N) \geq 1$. Let ϕ be an automorphism of $C_T(N)$. Then ϕ sends separating vertices to separating vertices.*

Proof. By Lemma 4.2.5, ϕ induces an isomorphism on an adjacency graph and the graph isomorphism maps cut points to cut points. Hence, ϕ must send separating vertices to separating vertices. □

Lemma 4.3.2 *Suppose that $\kappa(N) \geq 1$ and let ϕ be an automorphism of $C_T(N)$. Then, ϕ takes two-sided nonseparating vertices with nonorientable complements to two-sided nonseparating vertices with nonorientable complements in $C_T(N)$.*

Proof. Firstly, we note that the image of a two-sided nonseparating vertex under the automorphism ϕ is not a separating vertex, because ϕ sends a noncut point to a noncut point in some pants adjacency graph.

Let a be a two-sided nonseparating vertex such that $N \setminus a$ is nonorientable. When $\kappa(N)$ is at least four, we can find a $P - S$ decomposition $\mathbf{P}$ extending a where a corresponds to a vertex in the adjacency graph of $\mathbf{P}$ of valence three or four. Since ϕ induces an isomorphism on the adjacency graph, $\phi(a)$ must have the same valence. Since 2-separating vertices correspond to vertices of valence at most two, $\phi(a)$ can only be a nonseparating with nonorientable complement. Moreover, since two-sided nonseparating vertices with orientable complements correspond to vertices of valence one, $\phi(a)$ is a two-sided nonseparating vertex with nonorientable complements. □

Lemma 4.3.3 *Suppose that $\kappa(N) \geq 1$. Let ϕ be an automorphism of $C_T(N)$. Then ϕ takes 2-separating vertices to 2-separating vertices in $C_T(N)$.*

Proof. This lemma holds already when the boundary of N is at most one.

Let a be a 2-separating vertex in $C_T(N)$. Then $\phi(a)$ cannot be a separating or a nonseparating with nonorientable complement by Lemma 4.3.1 and Lemma 4.3.2. Assume that $\phi(a)$ is a two-sided nonseparating vertex with orientable complement. If the complexity of N is at least four then we extend a to a $P - S$ decomposition $\mathbf{P}$ where two two-sided nonseparating vertices with nonorientable complements adjacent to a, say b and c. By Lemma 4.2.4, $\phi(a)$, $\phi(b)$ and $\phi(c)$ are adjacent in $\phi(\mathbf{P})$. It follows that its valence is two. However, a two-sided nonseparating vertex with orientable complement in some adjacency graph has valence one. Hence, $\phi(a)$ must be a 2-separating vertex in $C_T(N)$. □

It follows from Lemma 4.3.1, Lemma 4.3.2 and Lemma 4.3.3 that we have the following lemma:

Lemma 4.3.4 *Suppose that $\kappa(N) \geq 1$. Let ϕ be an automorphism of $C_T(N)$. Then, ϕ takes two-sided nonseparating vertices with orientable complements to two-sided nonseparating vertices with orientable complements in $C_T(N)$.*

4.4 The kernel $ker(N)$

Definition 4.4.1 *Let a and b be two isotopy classes of two-sided simple closed curves. If $i(a, b) = 1$ or $i(a, b) = 2$ with zero algebraic intersection, then we say that a and b have minimal intersection.*

Lemma 4.4.2 *Let a, b_1 and b_2 be distinct two-sided vertices in $V(N)$ such that $i(b_1, b_2) = 0$ or otherwise minimal intersection and $i(a, b_i) = 0$ $for\, i = 1, 2$. Assume f_1 and f_2 are two mapping classes in $MCG(N)$ such that $f_i(d) = \phi(d)$ for all d in $V(a) \cup V(b_1) \cup V(b_2)$. Then, $f_1^{-1} f_2$ is in $Ker(N)$.*

Proof.

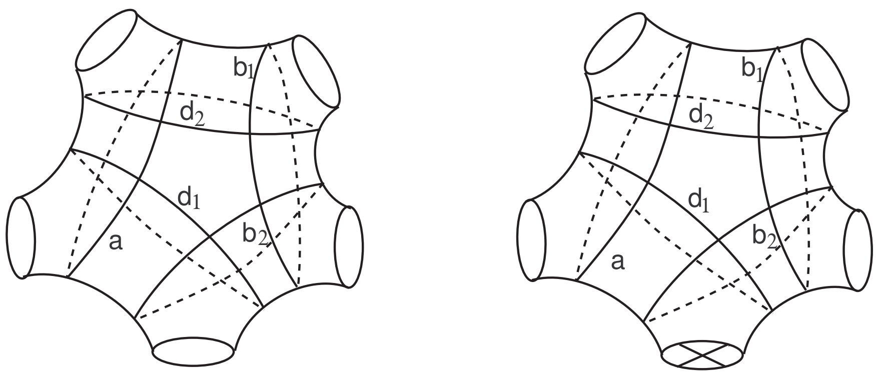

Figure 4.1: The case of N a five holed sphere on the left hand and the case of N a four holed real projective plane on the right hand.

Let f be the mapping class $f_1^{-1} f_2$, noting f acts trivially on $V(a)$. Assume that $f \notin ker(N)$. Then there are disjoint two-sided simple closed curves d_1 and d_2 on

N (possibly $d_1 = d_2$) such that at least one of $i(d_i, a)$ and $i(d_i, b_i)$ is zero, for both $i = 1, 2$, and such that $i(d_1, f(d_2))$ greater than zero. Moreover, we have $i(d_1, f(d_2))$, $i(d_1, f_1^{-1} f(d_2))$, $i(f_1(d_1), f_2(d_2))$ and $i(\phi(d_1), \phi(d_2))$ are equal to zero. We get a contradiction, and we prove the lemma.

Now, if we want to see that such a pair of curves d_1 and d_2 must exist, let us argue as follows. Assume d_1 is in $V(b_1)$ and it has minimal intersection with a and minimal intersection or zero intersection with b_2. In other words, d_1 is the only two-sided curve on N crossing a and disjoint from each curve in $V(d_1) \cap (V(a) \cup V(b_2))$. For an example (see Figure 4.1) in the five holed sphere, in this case, Luo described a pentagon configuration in [15]. In the Figure 4.1 (the case of N a five holed sphere), d_1 is only two-sided curve on N disjoint from both $b_1, d_2 \in V(d_1) \cap (V(a) \cup V(b_2))$. As it happens, $V(d_1) \cap (V(a) \cup V(b_2) = \{b_1, d_2\}$. Similarly, we can find a two-sided curve d_1 in the Figure 4.1 (the case of N a four holed real projective plane).

Assume that $i(d_1, f(d_2))$ is zero for any two-sided curve d_2 is in $V(d_1) \cap (V(a) \cup V(b_2))$. Then, $f(V(d_1) \cap (V(a) \cup V(b_2))) \subseteq V(d_1) \cap (V(a) \cup V(b_2))$. But, since f is a mapping class this inclusion is an equality and then we obtain $f(d_1) = d_1$. As the complement in N of b_1 is filled by a set of two-sided curves all fixed by f, we conclude f acts trivially on $V(b_1)$. Similarly, by reinterpreting our argument as $i(d_2, f^{-1}(d_1))$ is equal to zero we obtain f acts trivially on $V(b_2)$ as well. We have proved that $f(d) = d$ for all d is in $V(a) \cup V(b_1) \cup V(b_2)$. On the other hand, $V(a) \cup V(b_1) \cup V(b_2)$ fills N, that is each two-sided curve on N has nonzero intersection with some two-sided curve from this set. We obtain that f fixes each two-sided curve on N. Hence, f is in $Ker(N)$. $\square$

REFERENCES

[1] F. Atalan and M. Korkmaz, *Automorphisms of the curve complex on nonorientable surfaces*, Group Geom. Dyn. **8** (2014) 39-68.

[2] F. Atalan, *An algebraic characterization of a Dehn Twist for nonorientable surfaces*, arXiv : 1501.07183 .

[3] F. Atalan and B. Szepietowski, *Automorphisms of the mapping class group of a nonorientable surface*, Geometriae Dedicata, DOI. 10.1007 /s 10711-016-0216-7.

[4] S. C. Carlson, *Topology of Surfaces, Knots and Manifolds*, John Wiley and Sons, Inc. (2001).

[5] M. Dehn, *Papers on Group Theory and Topology*, (J. Stillwell (ed.)), Springer, New York, (1987).

[6] D. S. Dummit and R. M. Foote, *Abstract Algebra*, John Wiley and Sons, Inc. (2004).

[7] B. Farb and D. Margalit, *A Primer on Mapping Class Groups*, Princeton University Press, (2012).

[8] W. J. Harvey, *Geometric structures of surface mapping class group, in Homological Group Theory (C. T. Wall, ed.)*, London Mathematical Society Lecture Notes, No. 36, Cambridge University Press, London, 1979, pp. 255-269.

[9] N. V. Ivanov, *Mapping Class Groups*, preprint, (1998).

[10] N. V. Ivanov, *Automorphisms of complexes of curves and of Teichmuller spaces*, International Mathematics Research Notices (1997) 651-666.

[11] L. C. Kinsey, *Topology of Surfaces*, Springer-Verlag, New York, Inc. (1993).

[12] M. Korkmaz, *Minimal generating sets for the mapping class group of a surface*, in Handbook of Teichmüller spaces III. Editor: A. Papadopoulos, European Mathematical Society, (2012), 441-463.

[13] M. Korkmaz, *Automorphisms of complex of curves and on punctured spheres and on punctured tori*, Topology and its Applications, **95** (1999) 85-111.

[14] E. Kuno, *Abelian subgroups of the mapping class groups for non-orientable surfaces*, preprint, arXiv :1608.04501 (2016)

[15] F. Luo, *Automorphisms of the pants complexes of curves*, Topology, **39** (2000) 283-298.

[16] R. C. Penner and j. L. Harer, *Combinatorics of Train Tracks*, Princeton University Press, (1992).

[17] P. Schmutz Schaller, *Mapping class groups of hyperbolic surfaces and automorphism groups of graphs*, Composito Mathematica, **122** (2000) 243-260.

[18] K. J. Shackleton, *Combinatorial rigidity in curve complexes and mapping class groups*, Pacific Journal Of Mathematics, **230** (2007).