CONNECTEDNESS OF THE CUT-SYSTEM COMPLEX ON SURFACES

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
ATILIM UNIVERSITY

BY

FATEMA ALI

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS

MAY 2017

Approval of the Graduate School of Natural and Applied Sciences, Atılım University.

Prof. Dr. İbrahim Akman
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of **Master of Science in Mathematics Department, Atılım University**.

Prof. Dr. Tanıl Ergenç
Head of Department

This is to certify that we have read the thesis CONNECTEDNESS OF THE CUT-SYSTEM COMPLEX ON SURFACES submitted by FATEMA ALI and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Ferihe Atalan Ozan
Supervisor

Examining Committee Members:

Assoc. Prof. Dr. Sinem ONARAN
Mathematics Department, Hacettepe University

Assist. Prof. Dr. S. Öykü YURTTAŞ
Mathematics Department, Dicle University

Assoc. Prof. Dr. Ferihe ATALAN OZAN
Mathematics Department, Atılım University

Date: May 23, 2017

I declare and guarantee that all data, knowledge and information in this document has been obtained, processed and presented in accordance with academic rules and ethical conduct. Based on these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name : FATEMA ALI

Signature :

ABSTRACT

CONNECTEDNESS OF THE CUT-SYSTEM COMPLEX ON SURFACES

Ali, Fatema

M.S., Department of Mathematics

Supervisor : Assoc. Prof. Dr. Ferihe Atalan Ozan

May 2017, 28 pages

Let M be a compact, connected, orientable or nonorientable surface of genus $g \geq 1$ with n boundary components. In this thesis, we study connectedness of cut-system complex of M. More precisely, in Chapter 3, we study the work of Wajnryb on the connectedness of the cut-system complex of an orientable surface. In the last chapter, we prove that the cut-system complex of a nonorientable surface is connected.

Keywords: curves, cut-system complexes, surfaces

ÖZ

YÜZEYLER ÜZERİNDEKİ KESİM-SİSTEMİ KOMPLEKSİNİN BAĞLANTILILIĞI

Ali, Fatema

Yüksek Lisans, Matematik Bölümü

Tez Yöneticisi : Doç. Dr. Ferihe Atalan Ozan

Mayıs 2017, 28 sayfa

M kompakt, bağlantılı, cins sayısı $g \geq 1$ ve n sınır bileşenli yönlendirilebilen veya yönlendirilemeyen bir yüzey olsun. Bu tezde, M yüzeyinin kesim sistemi kompleksinin bağlantılılığını çalışacağız. Daha açık olarak söylersek, üçüncü bölümde, Wajnryb'ın yönlendirilebilen yüzeyin kesim sistemi kompleksinin bağlantılılığını ispat ettiği çalışmasını inceleyeceğiz. Son bölümde ise yönlendirilemeyen bir yüzeyin kesim sistemi kompleksinin bağlantılı olduğunu ispat edeceğiz.

Anahtar Kelimeler: eğriler, kesim sistemi kompleksleri, yüzeyler

To my mother, my father, my brother Mohammed, my husband, and my childern (Yousef and Farah)

ACKNOWLEDGMENTS

First of all, I am thankful to **Allah** for all the gifts has given me.

I shall also thank **my country** because it Damty my life in the scientific and practical.

I would like to thank my supervisor **Dr. Ferihe Atalan** for his continuous guidance, support and encouragement throughout my master studies and she worked with me.

I would like to thank also **Dr. Mehmet Turan** for answering my questions about graduate program in mathematics.

Finally, I would like to express my deepest gratitude to **my mother Aisha, my father Dhaw, my grand mother, my brothers, my sisters, my husband Faraj and my childern** for their endless support, encouragement and believing in me.

TABLE OF CONTENTS

LIST OF FIGURES

FIGURES

LIST OF SYMBOLS

M	:	An orientable or nonorientable surface
X	:	The cut-system complex
$P = (u_1, u_2, \cdots, u_k)$	:	A path
$\overline{M}$	:	The closed surface of M
$H_1(\overline{M}, \mathbb{Z})$	:	The homology group of $\overline{M}$ with $\mathbb{Z}$
$i(\alpha, \beta)$	:	The geometric intersection number
$\hat{i}(a, b)$	:	The algebraic intersection number
$[a]$	:	The homology class of a simple closed curve a

CHAPTER 1

INTRODUCTION

Let M be a compact, connected, orientable or nonorientable surface of genus $g > 0$ with $n \geq 0$ boundary components. We consider collections of g disjoint nonseparating simple closed curves $c_1, c_2, \ldots, c_g$ on the surface M, whose complement $M - (c_1 \cup \cdots \cup c_g)$ is a sphere with $2g + n$ boundary components. An isotopy class of these collections is called a cut-ststem. Cut-systems are the central objects of this thesis. Let $\langle c_1, c_2, \ldots, c_g \rangle$ be a cut-system. Assume that for some k, c_k' is a nonseparating simple closed curve transversely intersecting c_k at exactly one point and disjoint from c_i for $i \neq k$. Then if we replace c_k by c_k' in the cut-system, we obtain another cut-system. The operation of replacing curves is called an elementary move. There are three special types of paths which is described in Section 2. These special types of paths play an important role in the construction of the cut-system complexes. This cut-system complex is introduced by Hatcher and Thurston [7]. Therefore, this complex is also called Hatcher-Thurston complex in the literature.

The cut-system complex of surface M is a cell complex of dimension 2. Each cut-system is a 0-cell (vertex) of this complex. If two cells are related by an elementary move then these two 0-cells are joined by a 1-cell (an unoriented edge) corresponding to this move. Now, we have a graph, in other words; a 1-dimensional cell complex containing 0-cell and 1-cells. Let us attach 2-cells to this graph along boundaries resulting from the three special types of path. Then, we obtain the complex.

Hatcher and Thurston defined this complex for orientable surfaces. Their aim was to find a presentation for the mapping class group of an orientable surface. That's why there is an important place of this complex in the theory of mapping class groups. Moreover, Wajnryb used this complex to give a presentation for the mapping class group of an orientable surface and Harer also used it to compute the second homology group of the mapping class group in [5].

This complex is connected and simply-connected. More precisely, Hatcher and Thurston proved these results in [7]. Later, Wajnryb proved the same result by elementary techniques in [9].

The aim of this thesis is to study the connectedness of the cut-system complex (the Hatcher-Thurston complex). Our study has two components. First, in Chapter 3, we study connectedness of the cut-system (the Hatcher-Thurston) complex for orientable surfaces Wajnryb's result [9]. Second, in Chapter 4, we prove that the cut-system complex of a nonorientable surface is connected. We notice that the cut-system complex of a nonorientable surface is defined in [1]. We also note that we give the required definition and preliminary information used in Chapter 2.

CHAPTER 2

PRELIMINARIES AND NOTATIONS

In this chapter, we will give some definitions and notations related to surfaces, curves on surfaces, homology intersection numbers of curves and cut-systems of surfaces.

2.1 Surfaces and Curves on Surfaces

Definition 2.1.1 *(see pg 67, [8]) A topological space is called an n-dimensional manifold if each point has a neighborhood homeomorphic to an n-dimensional open disc. We also ask that any two distinct points have disjoint open neighborhoods and the topology has a countable basis.*

Definition 2.1.2 *(see pg 67, [8]) A* 2*-dimensional manifold is said to be a surface.*

Definition 2.1.3 *(see pg 70, [8]) A nonorientable surface is one which contains a Möbius band.*

Example 2.1.4 *The sphere, the torus and a closed orientable surface of genus g are examples of orientable surfaces.*

Example 2.1.5 *The Möbius band and the Klein bottle are nonorientable surfaces.*

Definition 2.1.6 *(see pg 31, [3]) A simple closed curve is a connected compact* 1*-dimensional manifold.*

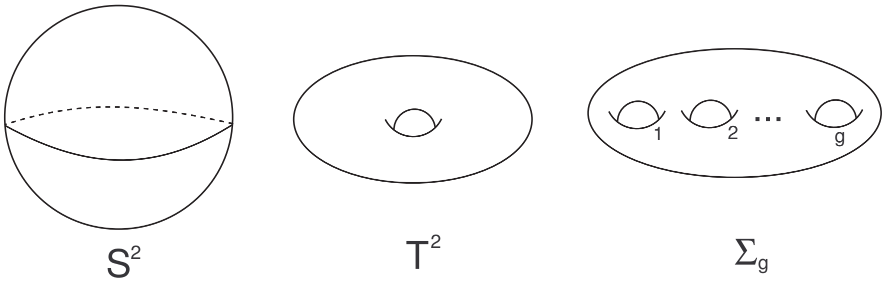

Figure 2.1: The sphere, the torus and a closed orientable surface of genus g

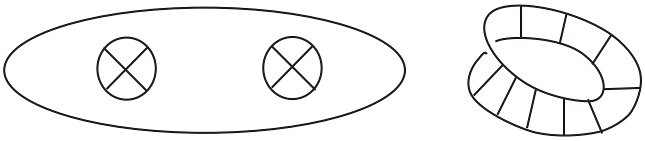

Figure 2.2: The Klein Bottle and the Möbius band

Definition 2.1.7 *Let us cut the surface M along that curve a. If you obtain two disconnected pieces of the surface M, the curve a is called a separating simple closed curve. Otherwise, it is a nonseparating simple closed curve.*

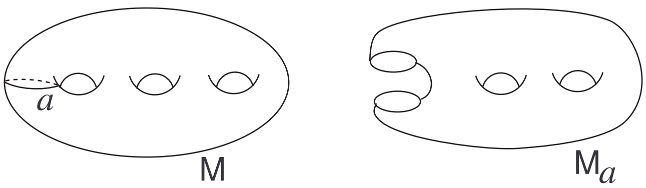

Figure 2.3: a is a nonseparating simple closed curve on the surface M.

Definition 2.1.8 *Let a be a simple closed curve. If its regular neighborhood is a Möbius band, then we say that a is one-sided. If its regular neighborhood is an annulus, then we say that a is two-sided.*

In particular, let a be a one-sided simple closed curve and let the genus of the surface M be $g = 1$ or $g \geq 2$. Let us cut the surface M along that curve a, if we get a

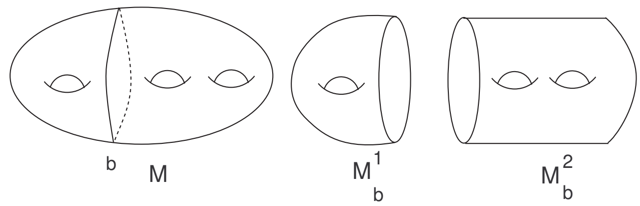

Figure 2.4: b is a separating simple closed curve on the surface M.

nonorientable surface, the curve a is called a one-sided essential simple closed curve.

Definition 2.1.9 *A simple closed curve a on a surface M is called trivial if the curve a bounds a disc, a disc with one puncture, a Möbius band or is homotopic to some boundary component of M.*

Definition 2.1.10 *(see pg 197, [8]) Let X and Y be two topological spaces and let f and g be two continuous functions from X to Y. Then f is homotopic to g if there is a continuous family of continuous functions $f_t : X \to Y$ for $0 \leq t \leq 1$, called a homotopy from f to g, satisfying*

- $f_0 = f$
- $f_1 = g$
- $f_t(x)$ *is continuous as a function of the pair* (x, t), $x \in X$, $t \in [0, 1]$.

Note that if each f_t is an embedding, then f is called an isotopy.

2.2 Homology group of a topological space X

Definition 2.2.1 *(see pg 64, [10]) Let X be a topological space. A (singular) n-simplex in X is a continuous map $\sigma : \Delta^n \to X$, where Δ^n is the standard n-simplex.*

A singular 0-simplex is identified with a point in X, because Δ^0 is a one point set. A singular 1-simplex is a path in X because Δ^1 is a closed interval.

Definition 2.2.2 *(see pg 64, [10]) Let X be a topological space. Let us define $S_n(X)$ as the free abelian group with basis all singular n-simplexes in X, for each $n \geq 0$. Let us define $S_{-1}(X) = 0$. The elements of $S_n(X)$ are said to be (singular) n-chains in X.*

We define the oriented boundary of a singular n-simplex $\sigma : \Delta^n \to X$ as

$$\sum_{i=0}^{n}(-1)^i(\sigma|_{[t_0,\ldots,\hat{t}_i,\ldots,t_n]}).$$

However, $[t_0, \ldots, \hat{t}_i, \ldots, t_n]$ is not the standard $(n-1)$-simplex and to fix this we identify this face with the standard $(n-1)$-simplex via the affine map

$$\epsilon_i = \epsilon_i^n : \Delta^{n-1} \to \Delta^n$$

defined by the formulas

$$\epsilon_0^n(u_0, \ldots, u_{n-1}) = (0, u_0, \ldots, u_{n-1}),$$

and

$$\epsilon_i^n(u_0, \ldots, u_{n-1}) = (u_0, \ldots, u_{i-1}, 0, u_i, \ldots, u_{n-1}),$$

for $i \geq 1$.

Example 2.2.3 *(see pg 64, [10]) There are three face maps $\epsilon_i^2 : \Delta^1 \to \Delta^2$:*

$$\epsilon_0 : [t_0, t_1] \to [t_1, t_2];$$

$$\epsilon_1 : [t_0, t_1] \to [t_0, t_2];$$

$$\epsilon_2 : [t_0, t_1] \to [t_0, t_1].$$

Definition 2.2.4 *(see pg 64, [10]) If $\sigma : \Delta^n \to X$ is continuous and $n > 0$, then its boundary is*

$$\partial_n\sigma = \sum_{i=0}^{n}(-1)^i\sigma\epsilon_i^n \in S_{n-1}(X).$$

We define $\partial_0\sigma = 0$ for $n = 0$.

We notice that if $X = \Delta^n$ and $\delta : \Delta^n \to \Delta^n$ is the identity, then

$$\partial(\delta) = \sum_{i=0}^{n} (-1)^i \epsilon_i^n.$$

We have the following result X in [10]:

Fact 2.2.5 *(see pg 64, [10]) There exists a unique homomorphism $\partial_n : S_n(X) \to S_{n-1}(X)$ with $\partial_n\sigma = \sum_{i=0}^{n}(-1)^i \sigma\epsilon_i$ for each singular n-simplex σ in X, for $n \geq 0$.*

Definition 2.2.6 *(see pg 65, [10]) The homomorphisms $\partial_n : S_n(X) \to S_{n-1}(X)$ are said to be boundary operators. For every X, we construct a sequence of free abelian groups and homomorphisms*

$$\ldots \longrightarrow S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \longrightarrow \ldots \longrightarrow S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0} 0,$$

called the singular complex of X. We denote $(S_*(X), \partial)$.

Fact 2.2.7 *(see pg 65, [10]) Let $k < j$. Then the face maps satisfy*

$$\epsilon_j^{n+1}\epsilon_k^n = \epsilon_k^{n+1}\epsilon_{j-1}^n : \Delta^{n-1} \to \Delta^{n+1}.$$

Example 2.2.8 *(see pg 65, [10]) Let $\epsilon_2^4\epsilon_0^3$ maps*

$$t_0 \mapsto t_1 \mapsto t_1;\ t_1 \mapsto t_2 \mapsto t_3;\ t_2 \mapsto t_3 \mapsto t_4$$

(the image is the 2-face $[t_1, t_3, t_4]$ of Δ^4); $\epsilon_0^4\epsilon_1^3$ maps

$$t_0 \mapsto t_0 \mapsto t_1;\ t_1 \mapsto t_2 \mapsto t_3;\ t_2 \mapsto t_3 \mapsto t_4.$$

Suppose that $k < j$. Then the image of $\epsilon_j\epsilon_k$ is $(n-1)$-face of Δ^{n+1} obtained by deleting vertices t_j and t_k; when $k \geq j$, the image deletes vertices t_j and t_{k+1}.

Fact 2.2.9 *(see pg 65, [10]) We have $\partial_n\partial_{n+1} = 0$, for all $n \geq 0$.*

Definition 2.2.10 *(see pg 65, [10]) Let $Z_n(X)$ denote the group of n-cycles in X. Then, $Z_n(X)$ is the ker(∂_n). Let $B_n(X)$ denote the group of n-boundaries in X. Then, $B_n(X)$ is the im(∂_{n+1}).*

Remark 2.2.11 *(see pg 65, [10]) $Z_n(X)$ and $B_n(X)$ are subgroups of $S_n(X)$ for $n \geq 0$.*

As an immediate corollary of Fact 2.2.9, we have

Fact 2.2.12 *(see pg 66, [10]) For each space X and for each $n \geq 0$,*

$$B_n(X) \subset Z_n(X) \subset S_n(X).$$

Definition 2.2.13 *(see pg 66, [10]) For every $n \geq 0$; the nth (singular) homology group of a space X is defined as the quotient group*

$$H_n(X) = \frac{Z_n(X)}{B_n(X)} = \frac{ker(\partial_n)}{im(\partial_{n+1})}.$$

Definition 2.2.14 *(see pg 66, [10]) The coset $z_n + B_n(X)$, where z_n is an n-cycle, is said to be the homology class of z_n. Let us denote it as $[z_n]$.*

We have the following result (see in [10]):

Theorem 2.2.15 *(see pg 67, [10]) If two spaces M_1 and M_2 are homeomorphic, then $H_n(M_1) \equiv H_n(M_2)$ for all $n \geq 0$.*

Remark 2.2.16 *(see pg 67, [10]) Each homology group $H_n(M)$ is a topological invariant of the space M.*

2.3 Intersection Numbers of Curves in a Surface

Intersection of two simple closed curves can be defined in two different ways, signed and unsigned intersection. These are the algebraic intersection number and geometric intersection number, respectively. Let a and b be a pair of transverse, oriented, simple closed curves in a surface M.

The algebraic intersection number $\hat{i}(a, b)$ is defined as the sum of the indices of the intersection points of a and b, where the index of an intersection point is $+1$ when the orientation of the intersection agrees with the orientation of the surface M and is -1

otherwise. The algebraic intersection number $\hat{i}(a,b)$ depends only on the homology classes $[a]$ and $[b]$.

If we count the minimal number of unsigned intersection points between homotopy classes of simple closed curves, we obtain the geometric intersection number. The geometric intersection number between free homotopy classes α and β of simple closed curves in a surface M is defined to be the minimal number of intersection points between a representative curve in the class α and a representative curve in the class β:

$$i(\alpha,\beta) = min\,\{|a \cap b| : a \in \alpha, b \in \beta\}\,.$$

We note that the geometric intersection number $i(\alpha,\beta)$ is symmetric:

$$i(\alpha,\beta) = i(\beta,\alpha),$$

while the algebraic intersection number $\hat{i}(\alpha,\beta)$ is skew-symmetric:

$$\hat{i}(\alpha,\beta) = -\hat{i}(\beta,\alpha).$$

We also note that the algebraic intersection number is well defined on homology classes, whereas the geometric intersection number is well defined only on free homotopy classes.

We observe that $i(\alpha,\alpha) = 0$ for any homotopy class of simple closed curves α on an orientable surface. If the curve a separates M into two components, then for any curve b we get $\hat{i}(a,b) = 0$ and $i(a,b)$ is even. In general, i and $\hat{i}$ have the same parity.

Example 2.3.1 *(see pg 29, [4]) The nontrivial free homotopy classes of oriented simple closed curves in the torus T^2 are in bijective correspondence with primitive elements of $\mathbb{Z} \oplus \mathbb{Z}$. For two such homotopy classes (r,h) and (r',h'), we have*

$$\hat{i}((r,h),(r',h')) = rh' - r'h$$

and

$$i((r,h),(r',h')) = |rh - r'h|.$$

2.4 Cut-systems on a Surface M

Let M be a compact orientable surface of genus g with n boundary components.

Definition 2.4.1 *Let us consider collections of g disjoint nontrivial nonseparating simple closed curves $c_1, c_2, \cdots, c_g$ in the surface M such that when we cut the surface M along these curves, we obtain a sphere with $2g+n$ holes. Let $\langle \gamma_1, \gamma_2, \ldots, \gamma_g \rangle$ denote an isotopy class of the collection $\{c_1, c_2, \cdots, c_g\}$. $\langle \gamma_1, \gamma_2, \ldots, \gamma_g \rangle$ is called a cut-system.*

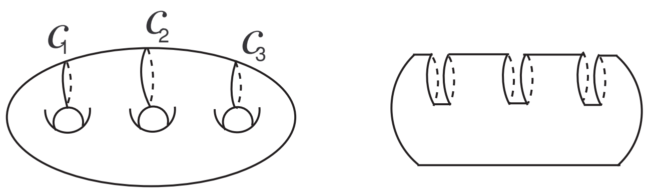

Figure 2.5: $\{c_1, c_2, c_3\}$ is a cut-system for genus 3 surface.

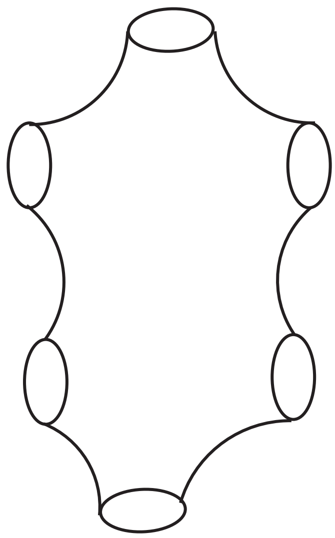

Figure 2.6: We obtain a sphere with six holes.

Definition 2.4.2 *Let $\langle \gamma_1, \gamma_2, \ldots, \gamma_{i-1}, \gamma_i, \gamma_{i+1}, \ldots \gamma_g \rangle$ be a cut-system and $c_i \in \gamma_i$ for all i. Let c_i' be a nontrivial nonseparating simple closed curve on the surface M such that* *it intersects c_i at one point and does not intersect other simple closed curves c_k for $k \neq i$ in the collection $\{c_1, c_2, ..., c_g\}$. If we change c_i by c'_i in the collection and we denote its isotopy class of c'_i by γ'_i, we get new a cut-system $\langle \gamma_1, \gamma_2, \cdots, \gamma_{i-1}, \gamma'_i, \gamma_{i+1}, \cdots, \gamma_g \rangle$. We call the replacement*

$$\langle \gamma_1, \gamma_2, \cdots, \gamma_{i-1}, \gamma_i, \gamma_{i+1}, \cdots, \gamma_g \rangle \longrightarrow \langle \gamma_1, \gamma_2, \cdots, \gamma_{i-1}, \gamma'_i, \gamma_{i+1}, \cdots, \gamma_g \rangle$$

an elementary move or a simple move.

We note that we will usually throw away the symbols for unchanged simple closed curves and so, we will simply write $\langle \gamma_i \rangle \longrightarrow \langle \gamma'_i \rangle$.

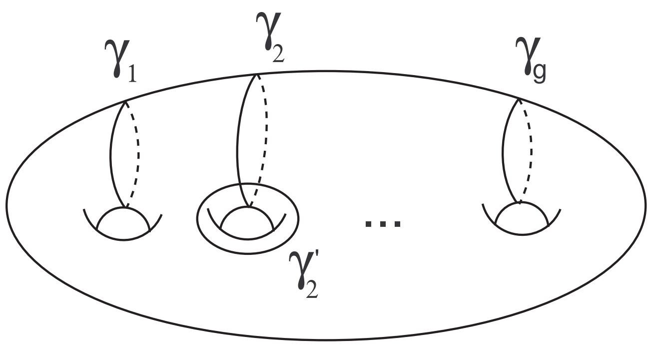

Figure 2.7: An example of an elementary move

2.4.1 Description of the Cut-system Complex

Let X denote the cut-system complex of a surface M. The vertices of the complex X are the cut-systems on the surface M. Two vertices are connected by an edge if and only if the corresponding cut-systems are related by an elementary move. Let X^1 denote the 1-skeleton of X. A path in X^1 is formed a sequence of vertices $P = (u_1, u_2, \cdots, u_k)$, where two consecutive vertices are related by an elementary move. If $u_1 = u_k$, then a path is closed. We have three types of closed paths in X^1.

Type 1. (A triangle) Suppose that three vertices have $(g - 1)$ simple closed curves in common. If the remaining three simple closed curves c_g, c'_g and c''_g intersect each other at one point, then the vertices constitute a closed triangular path:

$$\langle\gamma_g\rangle \to \langle\gamma'_g\rangle \to \langle\gamma''_g\rangle \to \langle\gamma_g\rangle$$

where γ_g, γ'_g and γ''_g are the isotopy classes of c_g, c'_g and c''_g, respectively.

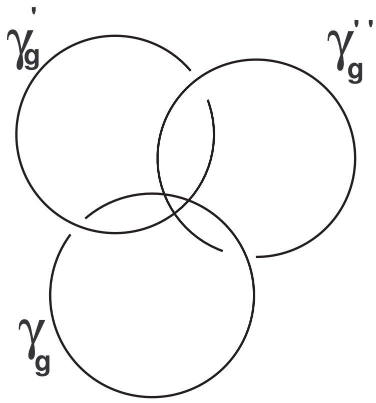

Figure 2.8: Configurations of curves in type (1)

Type 2. (A square) Suppose that four vertices have $(g-2)$ simple closed curves in common. If the remaining pairs of simple closed curves (c_{g-1}, c_g), (c'_{g-1}, c_g), (c'_{g-1}, c'_g), (c_{g-1}, c'_g) where these simple closed curves intersect as in figure. Then the vertices constitute a closed square path:

$$\langle\gamma_{g-1}, \gamma_g\rangle \to \langle\gamma'_{g-1}, \gamma_g\rangle \to \langle\gamma'_{g-1}, \gamma'_g\rangle \to \langle\gamma_{g-1}, \gamma'_g\rangle \to \langle\gamma_{g-1}, \gamma_g\rangle$$

where γ_{g-1}, γ_g, γ'_{g-1}, and γ'_g are the isotopy classes of c_{g-1}, c_g, c'_{g-1} and c'_g, respectively.

Type 3. (A pentagon) Suppose that five vertices have $(g-2)$ simple closed curves $c_1, c_2, \cdots, c_{g-2}$ in common. If the remaining pairs of simple closed curves (c_{g-1}, c_g), (c_{g-1}, c'_g), (c'_{g-1}, c'_g), (c'_{g-1}, c''_g) and (c_g, c''_g) where these simple closed curves intersect as in figure. Then the vertices constitute a closed pentagon path:

$$\langle\gamma_{g-1}, \gamma_g\rangle \to \langle\gamma_{g-1}, \gamma'_g\rangle \to \langle\gamma'_{g-1}, \gamma'_g\rangle \to \langle\gamma'_{g-1}, \gamma''_g\rangle \to \langle\gamma''_g, \gamma_g\rangle \to \langle\gamma_{g-1}, \gamma_g\rangle$$

where γ_{g-1}, γ_g, γ'_{g-1}, and γ'_g and γ''_g are the isotopy classes of c_{g-1}, c_g, c'_{g-1}, c'_g and c''_g, respectively.

If we attach a $2-$ cell to each closed edge-path of type (1), (2) or (3) in the 1-skeleton

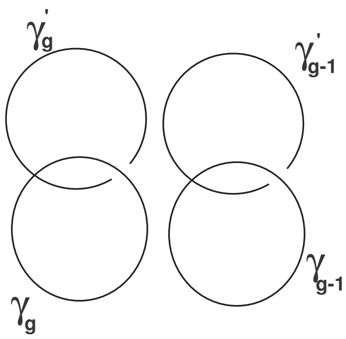

Figure 2.9: Configurations of curves in type (2)

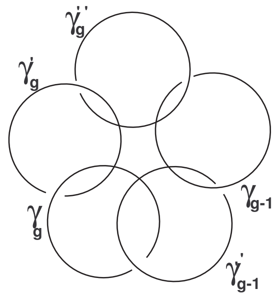

Figure 2.10: Configurations of curves in type (3)

X^1, then we get a $2-$ dimensional cell complex. We call this the cut-system complex, we denote it by X. This complex is defined by Hatcher and Thurston in 1980. Therefore, this is also called the Hatcher-Thurston complex in the literature.

CHAPTER 3

CONNECTEDNESS OF CUT-SYSTEM COMPLEX OF ORIENTABLE SURFACES

3.1 Introduction

Let M be a compact, connected orientable surface of genus $g > 0$ with $n \geq 0$ boundary components. Let $\overline{M}$ denote a closed orientable surface obtained from M by capping every boundary component with a disk. In this chapter, we study Wajnryb's work related to the connectedness of cut-system complex of an orientable surface. The main aim in the proofs in this section is to decrease the number of the intersection points between two distinct simple closed curves.

If the Euler characteristic of the surface M is negative, we consider a hyperbolic metric on the surface M. Therefore, the isotopy class of a nonseparating curve consists of a unique simple closed geodesic, the shortest curve in its isotopy class.

Definition 3.1.1 *(see pg 413, [9]) Let a and b be simple closed curves on a surface M. If there are curves a' and b', isotopic to a and b respectively, and such that $i(a, b) > i(a', b')$, we say that they have an excess intersection.*

Definition 3.1.2 *(see pg 413, [9]) Let a and b be simple closed curves on a surface M. If there are arcs α of a and β of b, which intersect only at their common end points and do not intersect other points of a or b and such that $\alpha \cup \beta$ bounds a disk (possibly with holes) on M, they form a 2-gon. The disk is said to be a 2-gon. Let us cut off the 2-gon from a by replacing the arc α of a by the arc β of b. We obtain a new curve a' (see Figure 3.1).*

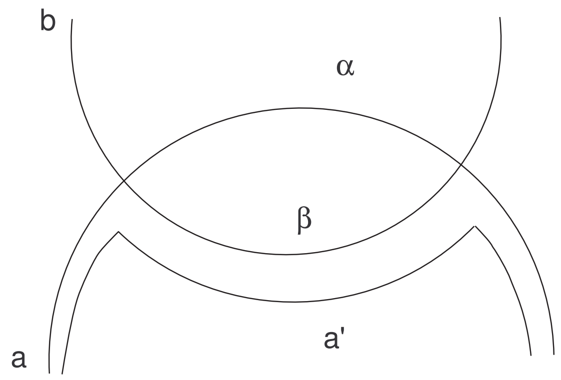

Figure 3.1: Curves a and b form a 2-gon.

We will give the following lemma proved by Hass-Scott (see [6], Lemma 3.1).

Lemma 3.1.3 *Let a, b be two simple closed curves on a surface M. If they have an excess intersection, then they form a* 2*-gon (without holes) on the surface M.*

Corollary 3.1.4 *Two simple closed geodesic on the surface M do not have excess intersection. In particular, suppose that we change two curves a and b by geodesics in their isotopy class. Then, $i(a, b)$ does not increase.*

Proof. Suppose that there exists a 2-gon. Let us cut off 2-gon and then smoothing corners, so we can decrease one simple closed curve in its homotopy class. We obtain the desired result. □

Let a and b be two simple closed curves in a finite collection of simple closed geodesics on an orientable surface M. Assume that a and b constitute a minimal 2-gon. This means that it does not consist of another 2-gon. Let us cut off the minimal 2-gon from a and pass to the isotopic geodesic, then we obtain the simple closed curve, say a'.

Lemma 3.1.5 *Let a, b and a' be as in above paragraph. Then, $i(a, a') = 0$, $i(b, a') < i(b, a)$ and $i(c, a') \leq i(c, a)$ for any other simple closed curve c of the collection. In particular, if $i(c, a) = 1$, then $i(c, a') = 1$. Moreover, $[a] = [a']$.*

Proof. Let α and β be two arcs of a and b, respectively; forming a minimal 2-gon. Since this 2-gon is minimal, each other simple closed curve c of the collection crosses the 2-gon along arcs crossing α and β once. Therefore, cutting off the 2-gon will not alter $i(a, c)$. After passing to the isotopic geodesic, this number may decrease. We note that a and a' are disjoint and homologous on the surface $\bar{M}$. By passing to a' at least two intersection points of a with b are removed. □

3.2 The case of genus $g = 1$

Let $[a]$ be the homology class of a simple closed curve a on a closed torus $\bar{M}$. Let us fix a basis of $H_1(\bar{M}, \mathbb{Z})$. Then, the homology class $[a]$ can be defined by a pair of relatively prime integers, up to a sign. Suppose that $[a] = (\alpha_1, \alpha_2)$ and $[b] = (\beta_1, \beta_2)$. The absolute value of their algebraic intersection number is equal $|\hat{i}(a, b)| = |\alpha_1\beta_2 - \alpha_2\beta_1|$. If a and b are geodesics on $\bar{M}$, then $i(a, b) = |\hat{i}(a, b)|$. Since they do not have excess intersection on $\bar{M}$, this is also true for curves on the surface M forming no 2-gons.

Lemma 3.2.1 *Let a and b be nonseparating simple closed curves on the surface M. Assume that $i(a, b) = k$, where $k \neq 1$. Then there is a nonseparating simple closed curve d such that*

- $i(d, a) = i(d, b) = 1$, *if* $k = 0$;
- $i(d, a) < k$ *and* $i(d, b) < k$, *if* $k > 1$.

Proof. There are two cases: $k = 0$ and $k > 1$.

First, assume that $i(a, b) = 0$. Then the simple closed curves a and b are isotopic on $\bar{M}$. They split the surface M into two connected components, say M_1 and M_2. We pick points R and H on the simple closed curves a and b, respectively. Let us join the points R and H by simple arcs in M_1 and in M_2. The union of these arcs constitutes the desired curve d.

Now, assume that $k > 1$. If the simple closed curves a and b have an excess intersection on $\bar{M}$, then they constitute a 2-gon on the surface M. Let us cut off the 2-gon. It follows from Lemma 3.1.5 that we decrease the intersection. On the other hand, if they do not form 2-gons, then the algebraic intersection number is equal to the geometric intersection number. We note that the sign of all intersections are the same. Let us consider two intersection points consecutive along b. Then we pick d as on Figure 3.2. So, we obtain that $i(d, a) = 1$ and d is a nonseparating simple closed curve such that $i(d, b) < k$. □

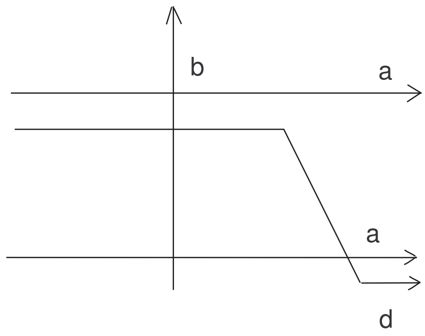

Figure 3.2: The simple closed curve d has smaller intersection with the simple closed curves a and b.

Proposition 3.2.2 *If surface M has genus one, then the cut-system complex X of the surface M is connected.*

Proof. Since the genus of the surface M is one, a cut-system on the surface M is an isotopy class of a single simple closed curve. If the intersection number of two simple closed curves is one, they are joined by an edge. Using Lemma 3.2.1, any two simple closed curves are joined by an edge-path in the complex X by induction. □

3.3 The case of genus $g > 1$

In this section, let M be a compact, connected, orientable surface of genus $g > 1$ with $n \geq 0$ boundary components. In this section, we will prove that the cut-system complex X of M is connected. We use induction on the genus of the surface. It is proved for the genus $g = 1$ in Section 3.2. Now, we assume that this is true for the genus less than g.

Lemma 3.3.1 *Let two vertices of the complex X have one or more simple closed curves in common. Then, they are connected by a path such that all of whose vertices consist of the common simple closed curves.*

Proof. Let us cut the surface M along the common curves. Then, the remaining collection of the simple closed curves constitute two vertices of the cut-system complex on the new surface. This new surface has smaller genus. So, they are joined by a path. If we take all the common curves to every vertex of this path we obtain a path in the complex X with the desired properties. □

Remark 3.3.2 *Let a and b be two disjoint distinct nonseparating simple closed curves on the surface M. The union of a and b does not separate the surface M if and only if their homology classes are not equal. In this case, we can complete a cut-system with a and b together on the surface M.*

Lemma 3.3.3 *Let c_1 and c_2 be nonseparating simple closed curves on the surface M such that $i(c_1, c_2) = k$, where $k \geq 2$. Then there is a nonseparating simple closed curve d such that $i(c_1, d) < k$ and $(c_2, d) < k$.*

Proof. Let us orient the simple closed curves c_1 and c_2. We split the union $c_1 \cup c_2$ into a different union of oriented simple closed curves as follows. We begin near an intersection point, say a point Q_1, on the side of the curve c_2 after the curve c_1 meets it and on the side of the curve c_1 before the curve c_2 intersects it. Now we go parallel to the curve c_1 to the next intersection point, say a point Q_2. We do not meet the curve c_2 at the point Q_2 and go parallel to the curve c_2, in the positive direction, back to the point Q_1. We obtain a curve d_1.

Now, we begin near the point Q_2 and go parallel to the curve c_1 until we cross an intersection point, say Q_3, which is either equal to the point Q_1 or was not crossed before. We do not meet the curve c_2 at the point Q_3 and go parallel to the curve c_2, in the positive direction, back to the point Q_2. We obtain a curve d_2. And so on. Curve d_i intersects the curve c_1 near some points of the intersection of the curves c_1 and c_2, but not near the point Q_i and it intersects the curve c_2 near some points of the intersection of the curves c_1 and c_2, but not near the point Q_{i+1}. Hence, the curve d_i intersects

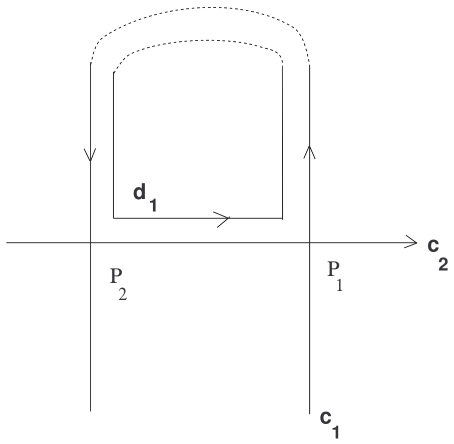

Figure 3.3: A curve d_1

both curves less than k-times. If c is an oriented simple closed curve in $H_1(\bar{M}, \mathbb{Z})$, its (oriented) homology class is denoted by $[c]$. We have $[c_1] + [c_2] = [d_1] + \cdots + [d_s]$. Using a similar construction for the opposite orientation of the curve c_2 beginning near the same point Q_1. We obtain new curves $e_1, e_2, \ldots, e_r$ and $[c_1] - [c_2] = [e_1] + \ldots + [e_r]$. We also have $[d_1] - [e_1] = [c_2]$. If we combine these equalities in $H_1(\bar{M}, \mathbb{Z})$, we obtain $[e_1] + \sum_{i \neq 1} [d_i] = [c_1]$, $[d_1] + \sum_{i \neq 1} [e_i] = [c_1]$, $\sum_{i \neq 1} [d_i] - \sum_{i \neq 1} [e_i] = [c_2]$. A simple closed curve is separating on the surface M if and only if it represents 0 in $H_1(\bar{M}, \mathbb{Z})$. As the curves c_1 and c_2 are nonseparating, we obtain that either d_1 and some d_i, $i \neq 1$, are not separating or e_1 and some e_i, $i \neq 1$, are not separating. And each of them intersects the curves c_1 and c_2 less than k-times, hence, it can be chosen for the curve d. □

Lemma 3.3.4 *Let c_1 and c_2 be two distinct, disjoint nonseparating simple closed curves on the surface M such that their union separates the surface M. Then there is a nonseparating simple closed curve d such that $i(c_1, d) = 1$ and $i(c_2, d) = 1$.*

Proof. Since $c_1 \cup c_2$ separates the surface M, we have two connected components, say M_1 and M_2. Let us choose a simple arc δ_1 in M_1 connecting c_1 to c_2. Similarly, let us choose a simple arc δ_2 in M_2 connecting c_1 to c_2. Now, let us slide the end-points of δ_1 and δ_2 over c_1 and c_2 to meet the end-points. We obtain a nonseparating simple closed curve d intersecting c_1 and c_2 at one point. □

Lemma 3.3.5 *Let c_1 and c_2 be two disjoint nonseparating simple closed curves as in Lemma 3.3.4. Let a and b be also two another nonseparating simple closed curves and be given a positive integer n such that*

- $i(a, b) \leq n$,
- $i(a, b) = 1$ *if* $n = 1$,
- $i(c_1, a) < n$, $i(c_2, a) \leq n$, *and*
- $i(b, c_1) = i(b, c_2) = 0$.

Then there is a simple closed curve d as in Lemma 3.3.4 which also fulfills $i(d, a) < n$ and $i(d, b) < n$.

Proof. Assume that the curves a and b and a positive integer n are given. As in the proof of Lemma 3.3.4, we have arcs δ_1 and δ_2. If necessary, we need to change the simple arcs δ_1 and δ_2, to decrease the intersection of d with the curves a and b. We can suppose that the curve b lies in the component M_1. We have several cases.

Case 1. The curve a lies in the component M_1. So, δ_2 is disjoint from the curve a. Let $n = 1$. Then, we have $i(a, b) = 1$. So, $a \cup b$ does not separate its regular neighbourhood and does not separate the component M_1. Now, choose δ_1 disjoint from the curves a and b. So, d is disjoint from the curves a and b, too.

Let $n > 1$. If the curve a separates the component M_1 (does not separate the surface M) then it separates the curve c_1 from the curve c_2 in the component M_1. There is a simple arc δ in the component M_1 connecting the curve c_1 to the curve c_2 and is disjoint from the curve a, if the curve a does not separate the component M_1, or intersects the curve a at one point, if the curve a separates the component M_1. Let us choose such a simple arc δ having minimal number of intersections with the curve b. Assume that $i(b, \delta) > n$. Now, there are two points Q and H of the intersection the curve b and the arc δ, consecutive along the curve b, and not separated by a point of intersection the curve b and the curve a. We can go along the arc δ to the point Q then along the curve b, without crossing the curve b, to the point H. Then, we continue along the arc δ to its end. Now, we obtain a simple arc meeting the curve a

at most one point and having smaller number of intersections with the curve b. Thus, we can suppose that $i(b, \delta) = n$. Also, we can assume that each pair of points of the intersection of the curve b and the arc δ consecutive along the curve b is separated by a point of intersection of the curves b and a. Now, let us change the arc δ as follows. Let us take the intersection of the arc δ and the union $a \cup b$. If the initial or the final point along the arc δ of this intersection belongs to the curve a, then we begin from this end of the arc δ, or else, we begin from any end. We go along the arc δ to the initial point, say Q, of intersection with the union of the curves a and b. If the point Q on the curve a, we keep going along the curve a, without crossing it, to the next point of the intersection of the curves a and b. Then we go along the curve b, without crossing it, to the final point, say H, of the intersection of the curve b and the arc δ on the arc δ, and then along the arc δ to its end. Now, the new arc meets the curve b at most one point, near the point H, and meets the curve a less than n-times. If the point Q on the curve b, we keep going along the curve b, without crossing the curve b, to the point H. And then keep going along the arc δ to its end, which shows a similar result. We can select such an arc for the arc δ_1. So, d satisfies the lemma.

Case 2. The curve a intersects the curves c_1 or c_2. If $m = 1$ and $i(c_1, a) = 0$ and the curve a meets the curve c_2 in the component M_1, then it must intersect it again to exit the component M_1. However, it gives a contradiction with $i(c_2, a) \leq m$. Therefore, we have $m > 1$. The arcs of the curve a split the component M_1 (and the component M_2) into connected components. One of the components crosses both the curves c_1 and c_2 (or else, the union of all components crossing the curve c_1 contains the curve a for a boundary component. So, the curve a is disjoint from the curves c_1 and c_2). Let us select the arc δ_1 (respectively the arc δ_2) in such a component. Now, the curve a can be disjoint from them. We require to change the arc δ_1 in such a way that $i(\delta_1, a) = 0$ and $i(\delta_1, b) < m$. Now, we have three subcases.

Subcase a. There is a simple arc α_1 of the curve a in the component M_1 connecting the curves c_1 and c_2. Let us select the arc δ_1 parallel to the arc α_1. It can be that the arc δ_1 meeting the curve b n-times. Thus, the arc α_1 is the only arc of the curve a meeting the curve b. Then, we change the arc δ_1 as follows. We continue from the curve c_1 along the arc δ_1 until it intersects the curve b. Then we return along the curve b, away from the arc α_1, to the next point of the arc α_1. We return before meeting the arc α_1

and go parallel to the arc α_1 to the curve c_2. So, the new arc does not intersect the curve a and crosses the curve b less than n-times.

Subcase b. There is a simple arc of the curve a in the component M_1 connecting the curve c_1 and the curve b. There is also a simple arc of the curve a connecting the curve c_2 and the curve b. Then we have points Q and H of the intersection of the curves a and b, consecutive along the curve b, and arcs α_1 and α_2 of the curve a such that the arc α_1 joins the curve c_1 to the point Q and the arc α_2 joins the point H to the curve c_2. We go along the arc α_1 to the point Q, then, along the curve b, without meeting the curve b, to the point H, and then, along the arc α_2 to the curve c_2. So, the new arc does not intersect the curve a and intersects the curve b less than n-times.

Subcase c. If a simple arc of the curve a in the component M_1 intersects the curve b, then it intersects only the curve c_1. (The case of the curve c_2 is similar.) Let us take into account that a simple arc δ in the component M_1 disjoint from the curve a and connecting the curves c_1 and c_2. We begin at the curve c_2 and go along the arc δ to the initial point of intersection with the curve b. Then we go along the curve b, without meeting it, to the initial point of intersection with the curve a. Then we go along the curve a, away from the curve b, to the curve c_1. So, the new arc is disjoint from the curve a and intersects the curve b less than n-times. If $i(b, a) = 0$, then the curve b is either disjoint from a component of $M_1 \setminus a$ connecting the curve c_1 to the curve c_2 or is included in it. One can see an arc in the component (disjoint from the curve a) connecting the curve c_1 with the curve c_2 and intersecting the curve b at most one point.

We have a simple arc δ_1 disjoint from the curve a and intersecting the curve b less than n-times in every case. Now, let us slide the end points of the arc δ_1 along the curves c_1 and c_2 to cross the end points of the arc δ_2. Every slid can be made along one of two arcs of the curve c_i. Let us choose suitably the arc δ_1 meeting at most $\frac{n}{2}$ points of the curve a sliding along the curve c_2 and at most $\frac{(n-1)}{2}$ points of the curve a sliding along the curve c_1. We obtain the curve d from the arcs δ_1 and δ_2. It intersects the curves a and b less than n-times.

Case 3. The curve a is on the component M_2. Then, the curves a and b are disjoint. So, we must have $n > 1$. One can choose an arc δ_2 disjoint from the curve b and intersecting the curve a at most one point. One can also choose an arc δ_1 disjoint from the curve a and intersecting the curve b at most one point. We obtain the curve d from δ_1 and δ_2 and it intersects the curves a and b less than n-times. □

Lemma 3.3.6 *Let d_1 and d_2 be nonseparating simple closed curves on the surface M. Suppose that v_1 and v_2 are vertices of the complex X containing the curves d_1 and d_2, respectively. Then, there is an edge-path $t = (v_1 = Z_1, Z_2, \ldots, Z_k = v_2)$ connecting v_1 and v_2.*

Proof. We prove the lemma by induction on $i(d_1, d_2) = m$.

If the curves d_1 and d_2 are equal, then v_1 is connected to v_2, by Lemma 3.3.1.

Let $m = 1$. There are vertices w_1 and w_2 in the complex X connected by an edge and such that $d_1 \in w_1$ and $d_2 \in w_2$, respectively. Now, one can connect w_1 to v_1 and v_2 to w_2 as in the previous case.

Let $m = 0$ and the union $d_2 \cup d_1$ do not separate the surface M. Then there is a vertex u including both curves d_2 and d_1. Then, u is connected to v_1 and v_2 as in the first case.

Now, let $m = 0$ and the union $d_1 \cup d_2$ separate the surface M. By Lemma 3.3.4, there is a curve d such that $i(d_2, d) = 1$ and $i(d_1, d) = 1$. One can find a vertex u including the curve d and can connect u to v_1 and v_2 as in the second case.

Let $m > 1$. By Lemma 3.3.3, there is a curve d such that $i(d_1, d) < m$ and $i(d_2, d) < m$. Let us choose a vertex u including the curve d. By induction on m, u is connected to v_1 and v_2. □

Hence, we have the following corollary immediately:

Corollary 3.3.7 *The cut-system complex X of the surface M is connected.*

CHAPTER 4

CONNECTEDNESS OF THE CUT-SYSTEM COMPLEX OF NONORIENTABLE SURFACES

Let M be a compact, connected nonorientable surface of genus g with n boundary components. In this chapter, we study connectedness of the cut-system complex of a nonorientable surface. Firstly, we recall that the definition of the cut system and elementary move for nonorientable surfaces. Ashiba defined the cut-system and an elementary move for nonorientable surfaces as follows (see in [1]):

Definition 4.0.8 *Let a collection* $\{c_1, c_2, ..., c_g\}$ *be pairwise disjoint one-sided essential simple closed curves on the surface* M*. Let the collection of their isotopy classes denote by* $\langle \gamma_1, \gamma_2, \ldots, \gamma_g \rangle$. $\langle \gamma_1, \gamma_2, \ldots, \gamma_g \rangle$ *is said to be a cut-system if the surface obtained from* M *by cutting along all* c_i *in the collection is a sphere with* $g + n$ *boundary components.*

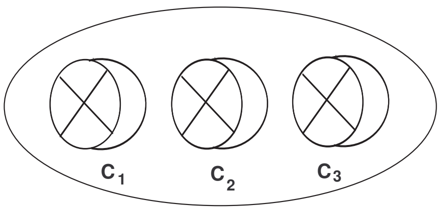

Figure 4.1: $\{c_1, c_2, c_3\}$ is a cut-system on a closed nonorientable surface of genus $g = 3$.

Definition 4.0.9 *Let* $\langle \gamma_1, \gamma_2, \ldots, \gamma_{i-1}, \gamma_i, \gamma_{i+1}, \ldots \gamma_g \rangle$ *be a cut-system on the surface* M*, and* $c_i \in \gamma_i$ *for all* i*. Let* c_i' *be a one-sided essential simple closed curve on the surface*

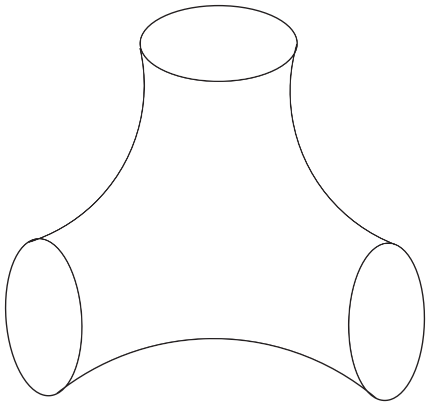

Figure 4.2: The sphere with three boundary components

M disjoint from c_k *for* $k \neq i$, $1 \leq i \leq g$ *and such that it intersects* c_i *at one point and does not intersect other simple closed curves in the collection* $\{c_1, c_2, ..., c_g\}$*. If we change* c_i *by* c'_i *in the collection, and we denote its isotopy class of* c'_i *by* γ'_i*, we get a new cut-system* $\langle \gamma_1, \gamma_2, \cdots, \gamma_{i-1}, \gamma'_i, \gamma_{i+1}, \cdots, \gamma_g \rangle$*. This operation is called an elementary move.*

We note that we will usually throw away the symbols for unchanged one-sided essential simple closed curves and thus, we will simply write $\langle \gamma_i \rangle \longrightarrow \langle \gamma'_i \rangle$*.*

Also, the complex of the cut-systems is described in a similar fashion to the orientable case (see [1]). Before we prove the main theorem, we will give the following proposition. This proposition is proved by Atalan and Korkmaz in [2].

Proposition 4.0.10 *Let M be a nonorientable surface of genus* $g \geq 1$ *with n boundary components. Let* d_1 *and* d_2 *be two one-sided essential simple closed curves on the surface M such that* $i(d_1, d_2) = k$*, where* $k \geq 2$*. There is a one-sided essential simple closed curve d such that* $i(d, d_1) < k$ *and* $i(d, d_2) < k$*.*

Remark 4.0.11 *Two distinct disjoint one-sided essential simple closed curves have different homology classes. Therefore, there is a cut-system containing these two one-sided essential curves.*

Theorem 4.0.12 *Let M be a compact, connected nonorientable surface of genus* $g \geq$

1 *with* $n \geq 0$ *boundary components. The cut-system complex is connected.*

Proof. We prove that this theorem by induction on the genus of the surface M.

Let $g = 1$. In this case, a cut-system (on the surface M) contains an isotopy class of a single curve. If two distinct two one-sided curves intersect at one point, we connect them by an edge. It follows from Proposition 4.0.10, by induction that any two one-sided essential curves can be joined by an edge path in the cut-system complex X.

Let $g \geq 2$. By induction hypothesis, we assume that the theorem holds for a nonorientable surface of genus less than g. We will prove that the complex X is connected for a nonorientable surface of genus g.

Let d_1 and d_2 be two one-sided essential simple closed curves on the surface M. Suppose that v_1 and v_2 are vertices of the complex X containing d_1 and d_2, respectively. We will show that there exists an edge path $P = (v_1 = Z_1, Z_2, \ldots, Z_k = v_2)$ connecting v_1 and v_2.

Case 1.

Let $d_1 = d_2$. Let us cut the surface M along this curve. The collection of the rest one-sided essential simple closed curves form two vertices of the cut-system complex on the obtained surface of smaller genus. By induction hypothesis, they can be connected by a path. If we take the curve to each vertex of this path we obtain a path in X. In fact, it is true that if two vertices of the complex X have one or more one-sided essential simple closed curve in common, then they are joined by a path all of whose vertices consist of the common one-sided essential curves.

Case 2.

Let $i(d_1, d_2) = 1$. Then, there are vertices w_1 and w_2 in the complex X which are joined by an edge and such that w_1 and w_2 contain d_1 and d_2, respectively. Then, we join w_1 to v_1 and v_2 to w_2 as in the above case.

Case 3.

Let $i(d_1, d_2) = 0$. Then, there is a vertex u containing both one-sided essential curves d_1 and d_2. So, the vertex u is connected to v_1 and v_2 as in the first case.

Case 4.

Let $i(d_1, d_2) = n > 1$. By Proposition 4.0.10 , there is a one-sided essential simple closed curve d such that $i(d_1, d) < n$ and $i(d_2, d) < n$. We pick a vertex u containing d. By induction on n, we connect the vertex u to v_1 and v_2, so the proof is complete. □

REFERENCES

[1] S. A. S. Ashiba, *The Hatcher-Thurston complex on a surface*, Ms. thesis, Atilim University, 2016.

[2] F. Atalan and M. Korkmaz, *Automorphisms of curve complexes on nonorientable surfaces*, Groups, Geometry, and Dynamics **8**, (2014) 39-68.

[3] S. C. Carlson, *Topology of Surfaces, Knots and Manifolds*, John Wiley and Sons Inc., (2001).

[4] B. Farb and D. Margalit, *A Primer on Mapping Class Groups*, Princeton University Press, (2012).

[5] J. Harer, *The second homology group of the mapping class group of a closed orientable surface*, Inventiones Mathematicae **72** (1983) 221-239.

[6] J. Hass and P. Scott, *Intersections of curves on surfaces*, Israel J. Math., **51**, (1985) 90-120.

[7] A. Hatcher and W. Thurston, *A presentation of the mapping class group of a closed orientable surface*, Topology **19** (1980) 221-237.

[8] L. C. Kinsey, *Topology of Surfaces*, Springer-Verlag, New York, Inc. (1993).

[9] B. Wajnryb, *An elementary approach to the mapping class group of a surface*, Geometry and Topology **3** (1999) 405-466.

[10] J. J . Rotman, *An Introduction to Algebraic Topology*, Springer-Verlag New York Inc., (1988).