THE HATCHER - THURSTON COMPLEX ON A SURFACE

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ABSTRACT

The Hatcher - Thurston Complex on a Surface

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In this thesis, we study the work of E. Irmak and M. Korkmaz on the automorphism group of the Hatcher-Thurston complex for a compact, connected, orientable surface of genus $g \geq 1$. More precisely, it is shown that this automorphism group is isomorphic to the extended mapping class group of the orientable surface modulo its center. In the last chapter of this thesis, we define cut systems and the Hatcher-Thurston complex for compact, connected, nonorientable surfaces of genus $g \geq 1$.

Keywords: Surfaces, Hatcher-Thurston complex, complex of curves

ÖZ

Bir Yüzey Üzerindeki Hatcher - Thurston Kompleksi

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Yüksek Lisans, Matematik Bölümü

Tez Yöneticisi : Doç. Dr. Ferihe Atalan Ozan

Nisan 2016, 29 sayfa

Bu tezde, kompakt, bağlantılı, yönlendirilebilen cins sayısı $g \geq 1$ olan bir yüzey için E. Irmak ve M. Korkmaz'ın Hatcher-Thurston kompleksinin otomorfizma grubu üzerindeki çalışmalarını inceleyeceğiz. Daha açık olarak, bu otomorfizma grubunun yönlendirilebilen yüzeyin genişletilmiş gönderim sınıf grubunun merkezine bölümüne izomorfik olduğu gerçeği üzerinde çalışılmaktadır. Bu tezin son bölümünde, cins sayısı $g \geq 1$ olan kompakt, bağlantılı, yönlendirilemeyen yüzeyler için Hatcher-Thurston kompleksini ve kesme sistemlerini tanımlayacağız.

Anahtar Kelimeler: Yüzeyler, Hatcher-Thurston kompleksi, eğrilerin kompleksi

To my mother, my father and my husband

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Σ	:	Orientable surface
N	:	Nonorientable surface
$Mod(\Sigma)$	:	The mapping class group of Σ
$Mod^*(\Sigma)$	:	The extended mapping class group of Σ
$HT(\Sigma)$	:	The Hatcher-Thurston complex of Σ
$C(\Sigma)$	:	The complex of curves of Σ
$Nonsep(\Sigma)$	:	The complex of nonseparating curves of Σ

CHAPTER 1

INTRODUCTION

Let Σ (respectively, N) be a closed, compact, connected, orientable (respectively, nonorientable) surface of genus g. Let S denote either a closed, compact, connected, orientable or nonorientable surface of genus g. The mapping class group $Mod(\Sigma)$ of an orientable surface Σ is defined as the group of isotopy classes of orientation-preserving diffeomorphisms $\Sigma \to \Sigma$. These groups play an important role in the low dimensional topology. There are several related groups, one of which is the extended mapping class group $Mod^*(\Sigma)$ of Σ defined as the group of the isotopy classes of all diffeomorphisms $\Sigma \to \Sigma$.

The complex of curves is one of the fundamental geometric objects on which the (extended) mapping class groups act. The complex of curves, the complex of nonseparating curves, the complex of separating curves, the complex of pants decompositions are some of these geometic objects. The complex of curves on orientable surfaces was discovered by W.J. Harvey [5] in 1979. The extended mapping class group $Mod^*(\Sigma)$ can be considered as the automorphism group of these geometric objects. Their automorphisms were first studied by N.V. Ivanov [11] in 1997. He proved that the group of automorphisms of the complex of curves $C(\Sigma)$ of Σ is isomorphic to the extended mapping class group of Σ, with the exception of a closed surface of genus two. In the genus two case, the map $Mod^*(\Sigma) \to AutC(\Sigma)$ is onto and its kernel is $\mathbb{Z}_2$ generated by the hyperelliptic involution. Moreover, he gave significant applications of this result to the mapping class groups and to the Teichmüller spaces. After his result, the automorphism groups of various other objects related surfaces has played an ever increasing role. P. Schmutz Schaller [17] studied the graph whose the set of vertices is the set of nonseparating simple closed curve geodesics and whose edges are pairs of vertices intersecting exactly once in 2000, D. Margalit [15] the complex of pants in 2004, T. E. Brendle - D. Margalit [2] the complex of separating simple closed curve in 2004. Also, some generalizations are made. M. Korkmaz [12] showed that Ivanov's result was extended to lower genus cases in 1999. F. Luo [14] proved Ivanov's result using different methods in all cases in 2000. F. Atalan - M. Korkmaz [1] extended this result to nonorientable surface in 2014. E. Irmak introduced a superinjective map of the complex of curves of Σ. She proved that a superinjective map of the complex of curves is induced by a mapping class of Σ in 2004 and 2006 [7], [8]. Using some of F. Atalan - M. Korkmaz's results, she also extended this result to nonorientable surfaces. Furthermore, E. Irmak showed that a superinjective map of the complex of nonseparating curves on an orientable surface Σ is induced by a diffeomorphisms of Σ.

In this thesis, we study one of the geometric objects mentioned above the Hatcher-Thurston complex on S. The vertices of the Hatcher-Thurston complex are cut systems. The cut systems are maximal sets of isotopy classes of nonseparating disjoint simple closed curves on S. E. Irmak - M. Korkmaz [10] proved that the group of automorphisms of the Hatcher-Thurston complex of an orientable surface Σ (for genus $g \geq 1$) is isomorphic to the extended mapping class group of Σ modulo its center in 2007.

This thesis is organized as follows. In chapter 2, we give the relevant definitions and preliminery information used in this work. In chapter 3, we study the automorphism group of the Hatcher-Thurston complex for orientable surfaces and E. Irmak - M. Korkmaz's result [10]. Finally, in chapter 4, we define cut systems and the Hatcher-Thurston complex for nonorientable surfaces of genus $g \geq 1$.

CHAPTER 2

Preliminaries and Notations

In this chapter, we will give basic definitions, some examples and preliminary information used in this thesis. Throughout this thesis, we consider only connected surfaces.

2.1 Surfaces

In this subsection, we introduce through definitions and examples, certain types of mathematical objects called manifolds. We are interested in 2-dimensional manifolds, that are, surfaces in this work.

Definition 2.1.1 *(see pg 28, [3]) A Hausdorff and second countable topological space is a 1-dimensional manifold if each of its points has a neighborhood which is homeomorphic to an open interval of the real line.*

Definition 2.1.2 *(see pg 28, [3]) A Hausdorff and second countable topological space is a 2-dimensional manifold if each of its points has a neighborhood which is homeomorphic to an open disk in* $\mathbb{R}^2$.

The examples of 1-dimensional manifolds are open intervals, unions of open intervals and the circle.

The simplest examples of 2-dimensional manifolds are $\mathbb{R}^2$ and subsets of $\mathbb{R}^2$ which are homeomorphic to open disks. The sphere, the torus and the Klein bottle are also 2-dimensional manifolds.

Definition 2.1.3 *(see pg 32, [3]) A surface is a connected 2-dimensional manifold.*

Definition 2.1.4 *(see pg 34, [3]) A compact surface is said to be nonorientable if it contains a subset which is homeomorphic to the Möbius band; otherwise, it is called orientable.*

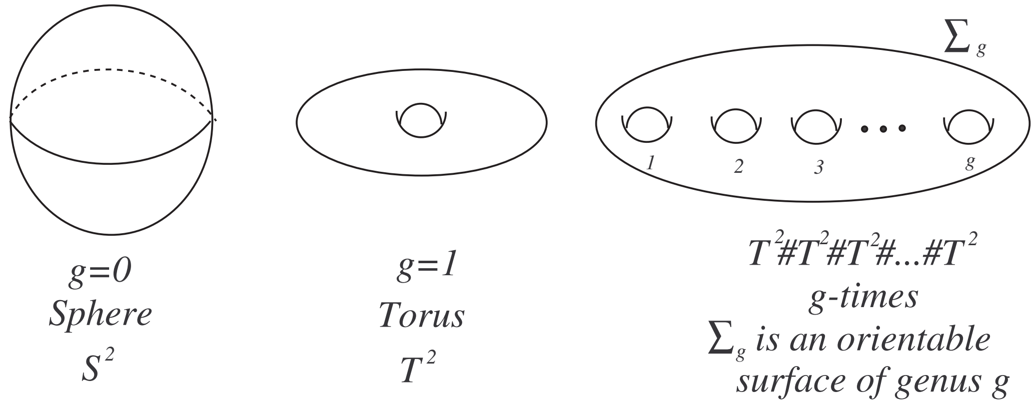

Figure 2.1: Orientable surfaces

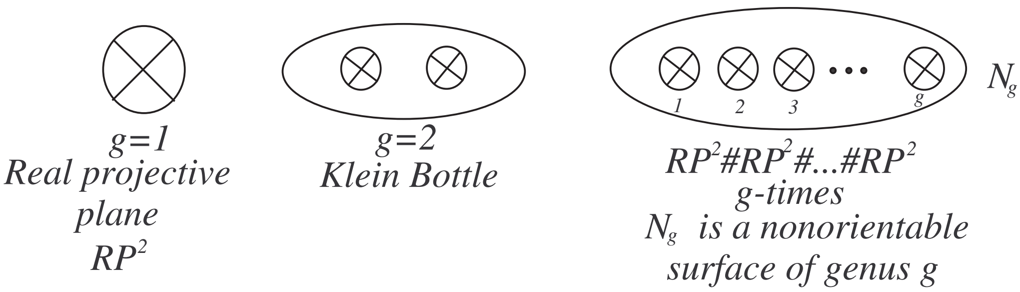

Figure 2.2: Nonorientable surfaces

2.2 Simple Closed Curves

Definition 2.2.1 *(see pg 31, [3]) A curve is a connected 1-dimensional manifold.*

Theorem 2.2.2 *(see pg 31, [3]) The Classification Theorem for Curves.*

(i) *Any compact curve is homeomorphic to the unit circle.*

(ii) *Any noncompact curve is homeomorphic to the real line.*

Let a be a simple closed curve on S. We call it a circle.

If a circle a bounds neither a disc nor a Möbius band on S, we say that a is a nontrivial circle. A trivial circle is a circle which is not nontrivial.

We say that a circle a is two-sided (respectively, one-sided) if a regular neighborhood of a is an annulus (respectively, a Möbius band).

Figure 2.3: The cylinder and the Möbius band

Let a be a circle. If we cut the surface S along the circle a,then the obtained surface is denoted by S_a. If S_a is a connected surface, then a is called nonseperating and separating otherwise.

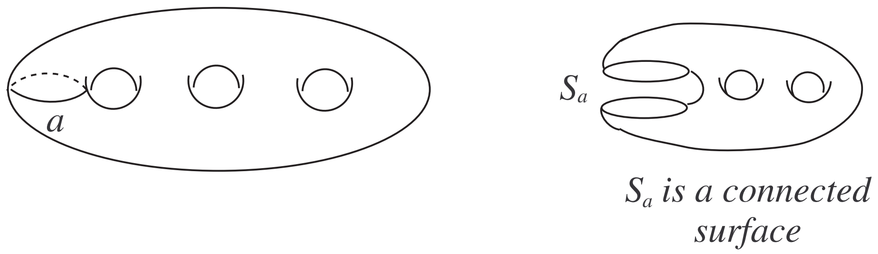

Figure 2.4: a is a nonseparating circle

We denote circles by the lowercase letters a, b, c and their isotopy classes by α, β, γ.

Let α and β be the isotopy classes of the two circles a and b, respectively; and let $|a \cap b|$ denote the cardinality of $a \cap b$. Then, we define the geometric intersection

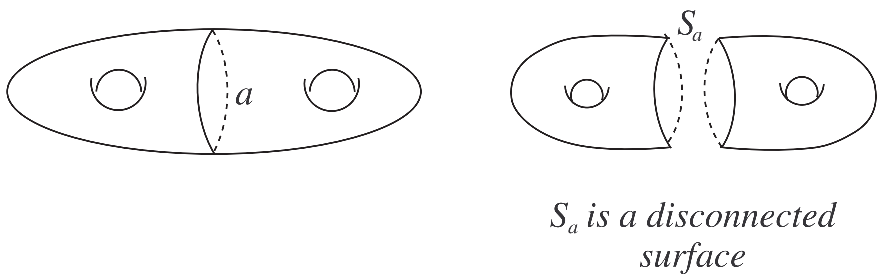

Figure 2.5: a is a separating circle

number $i(\alpha,\beta)$ as the minimum of $|a \cap b|$.

Any two circles are assumed to be intersect each other minimally. Any two circles are said to be dual on S, if they intersect each other transversely at only one point.

2.3 Cut Systems

Let a set $\{a_1, a_2, ..., a_g\}$ be pairwise disjoint nonseparating circles on Σ. Let the set $\left\langle \alpha_1, \alpha_2, ..., \alpha_g \right\rangle$ denote their isotopy classes. Let us cut the surface Σ along all a_i. Then if the resulting surface is connected, and so it is a sphere with $2g$ holes, we say that a set $\{a_1, a_2, ..., a_g\}$ is a geometric cut system on Σ. If $\{a_1, a_2, ..., a_g\}$ is a geometric cut system, then we say that the set of the isotopy classes $\left\langle \alpha_1, \alpha_2, ..., \alpha_g \right\rangle$ a cut system.

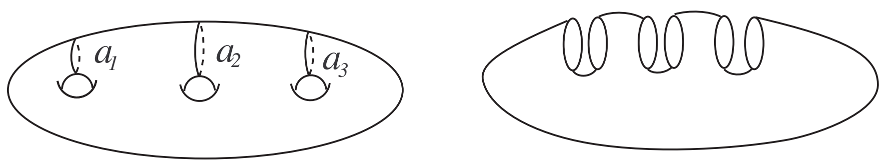

Figure 2.6: $\{a_1, a_2, a_3\}$ is a geometric cut system on an orientable surface

2.4 The Elementary Move

Let $\{c_1, c_2, ..., c_g\}$ be a geometric cut system on Σ. Assume that c' be a circle on Σ disjoint from circles c_j with $j \neq i$, $1 \leq i \leq g$, and dual to c_i. If we replace c_i by c' in the set $\{c_1, c_2, ..., c_g\}$, we obtain another cut system. An elementary move is the operation of replacing the geometric cut system $\{c_1, ..., c_i, ..., c_g\}$ by the geometric cut system $\{c_1, ..., c', ..., c_g\}$, and also the corresponding operation of replacing the cut system $\left\langle \gamma_1, ..., \gamma_i, ..., \gamma_g \right\rangle$ by the cut system $\left\langle \gamma_1, ..., \gamma', ..., \gamma_g \right\rangle$. We will usually throw away the unchanged circles and we will record $\langle \gamma_i \rangle \leftrightarrow \langle \gamma' \rangle$.

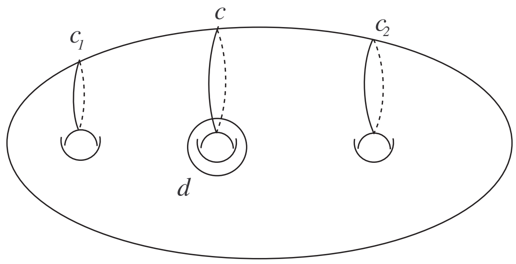

Figure 2.7: $\langle c_1, c, c_2 \rangle \leftrightarrow \langle c_1, d, c_2 \rangle$

2.5 The Hatcher-Thurston Graph

If we take cut systems on Σ as the set of vertices, then we obtain the Hatcher-Thurston graph of Σ. If one vertex is obtained from another by an elementary move, then these two vertices are connected by an (unordered) edge corresponding to this move. At this moment we have already the graph consisting of just described vertices and edges. It is denoted by $HT^1(\Sigma)$.

Let $(\gamma_1, ..., \gamma_n)$ be a sequence of cut systems. If each consecutive pair in $(\gamma_1, ..., \gamma_n)$ is joined by an edge, $(\gamma_1, ..., \gamma_n)$ forms a path in $HT^1(\Sigma)$. If $\gamma_1 = \gamma_n$, then we say that the path is closed. We have three types of special closed paths in the Hatcher-Thurston graph $HT^1(\Sigma)$.

1. **Triangles.** Let three vertices have $g - 1$ common isotopy classes. If the remain-

ing classes γ_i, γ_i', γ_i'' are dual pairwisely, then this closed path is a triangle. This triangle is denoted by $\langle\gamma_i\rangle \longleftrightarrow \left\langle\gamma_i'\right\rangle \longleftrightarrow \left\langle\gamma_i''\right\rangle \longleftrightarrow \langle\gamma_i\rangle$. In other words, let $\left\langle\gamma_1, ..., \gamma_i, ..., \gamma_g\right\rangle = \langle\gamma_i\rangle$, $\left\langle\gamma_1, ..., \gamma_i', ..., \gamma_g\right\rangle = \left\langle\gamma_i'\right\rangle$ and $\left\langle\gamma_1, ..., \gamma_i'', ..., \gamma_g\right\rangle = \left\langle\gamma_i''\right\rangle$. If γ_i, γ_i', γ_i'' satisfy $i(\gamma_i, \gamma_i') = i(\gamma_i, \gamma_i'') = i(\gamma_i', \gamma_i'') = 1$, then we have a triangle. (see Figure 2.9.)

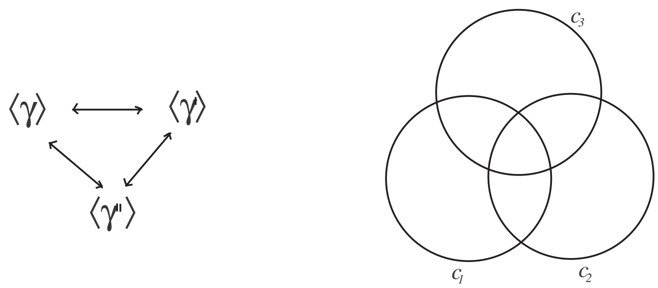

Figure 2.8: A triangle in the Hatcher-Thurston graph

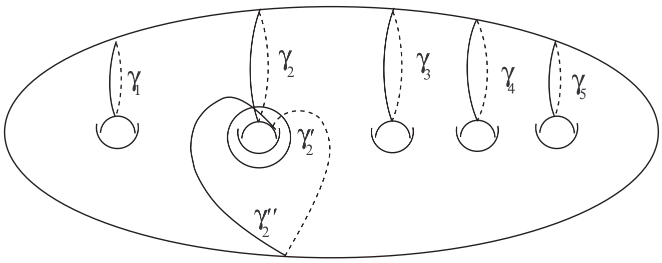

Figure 2.9: This figure is an example for the triangle in the Hatcher-Thurston graph on a closed orientable surface of genus 5, $\langle\gamma_2\rangle \longleftrightarrow \left\langle\gamma_2'\right\rangle \longleftrightarrow \left\langle\gamma_2''\right\rangle \longleftrightarrow \langle\gamma_2\rangle$.

2. **Rectangles.** Let four vertices have $g-2$ common isotopy classes. If the other classes γ_1, γ_2, δ_1, δ_2 have representatives c_1, c_2, d_1, d_2, respectively; as in Figure 2.10, then this closed path is a rectangle. More expilicitly,
$\left\langle\gamma_1, \delta_1, \beta_1 ..., \beta_{g-2}\right\rangle = \langle\gamma_1, \delta_1\rangle$, $\left\langle\gamma_1, \delta_2, \beta_1 ..., \beta_{g-2}\right\rangle = \langle\gamma_1, \delta_2\rangle$, $\left\langle\gamma_2, \delta_1, \beta_1 ..., \beta_{g-2}\right\rangle = \langle\gamma_2, \delta_1\rangle$ and $\left\langle\gamma_2, \delta_2, \beta_1 ..., \beta_{g-2}\right\rangle = \langle\gamma_2, \delta_2\rangle$ as in Figure 2.10, then the corresponding rectangle is denoted by $\langle\gamma_1, \delta_1\rangle \longleftrightarrow \langle\gamma_1, \delta_2\rangle \longleftrightarrow \langle\gamma_2, \delta_2\rangle \longleftrightarrow \langle\gamma_2, \delta_1\rangle \longleftrightarrow \langle\gamma_1, \delta_1\rangle$.

3. **Pentagons.** Let five vertices have $g-2$ common isotopy classes. If the other classes γ_1, γ_2, γ_3, γ_4, γ_5 have representatives c_1, c_2, c_3, c_4, c_5, respectively; as in

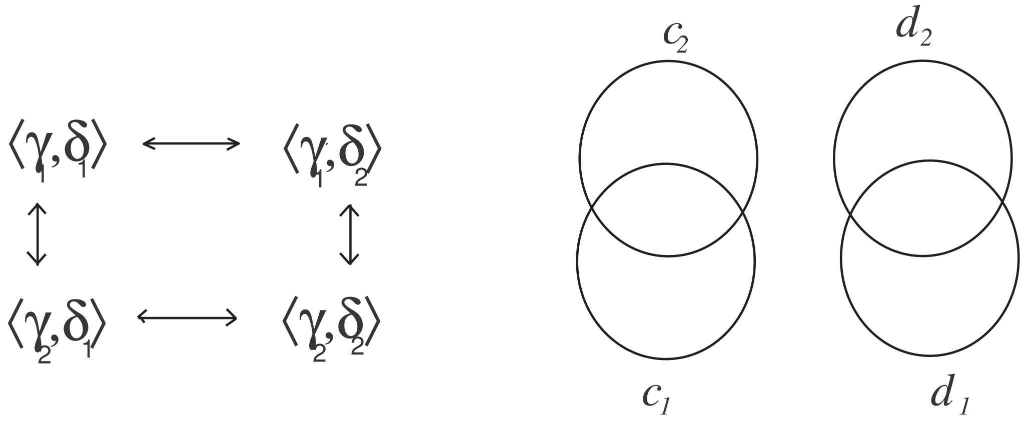

Figure 2.10: A rectangle in the Hatcher-Thurston graph

Figure 2.11, then this closed path is a pentagon. In other words, $\left\langle \gamma_1, \gamma_4, \delta_1, .., \delta_{g-2} \right\rangle = \langle \gamma_1, \gamma_4 \rangle$, $\left\langle \gamma_2, \gamma_4, \delta_1, .., \delta_{g-2} \right\rangle = \langle \gamma_2, \gamma_4 \rangle$, $\left\langle \gamma_2, \gamma_5, \delta_1, .., \delta_{g-2} \right\rangle = \langle \gamma_2, \gamma_5 \rangle$, $\left\langle \gamma_3, \gamma_5, \delta_1, .., \delta_{g-2} \right\rangle = \langle \gamma_3, \gamma_5 \rangle$, $\left\langle \gamma_1, \gamma_3, \delta_1, .., \delta_{g-2} \right\rangle = \langle \gamma_1, \gamma_3 \rangle$ as in Figure 2.11, then the corresponding pentagon is denoted by $\langle \gamma_1, \gamma_4 \rangle \leftrightarrow \langle \gamma_2, \gamma_4 \rangle \leftrightarrow \langle \gamma_2, \gamma_5 \rangle \leftrightarrow \langle \gamma_3, \gamma_5 \rangle \leftrightarrow \langle \gamma_1, \gamma_3 \rangle \leftrightarrow \langle \gamma_1, \gamma_4 \rangle$.

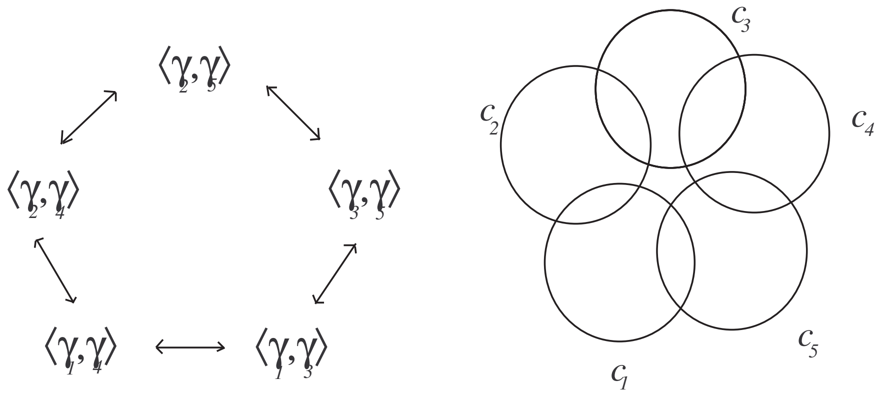

Figure 2.11: A pentagon in the Hatcher-Thurston graph

2.6 The Hatcher-Thurston Complex

If we attach a two cell along every triangle, rectangle and pentagon in the graph $HT^1(\Sigma)$, then we obtain a two dimensional CW-complex. We call this the Hatcher-Thurston complex $HT(\Sigma)$ of Σ.

In 1980, Hatcher and Thurston proved this complex is connected and simply connected for a compact, orientable surface of genus $g \geq 1$ ([6]). They gave a presentation of the mapping class group of a closed orientable surface by using this complex. In 1983, Wajnryb also used this complex to obtain a simple presentation for the mapping class group. Moreover, in 1999, Wajnryb gave another proof of the connectivity and simple connectivity of the complex $HT(\Sigma)$ ([18], [19]). More precisely, we have the following result shown in [6], [19]:

Theorem 2.6.1 *Let Σ be a compact, connected, orientable surface of genus $g \geq 1$. Then the complex $HT(\Sigma)$ is connected and simply connected.*

2.7 Various Complexes on Curves

First, we define an abstract simplicial complex (see [16]). Let V be a nonempty set. An abstract simplicial complex K with vertices V is a collection of nonempty finite subsets of V, called simplices, such that the followings are satisfied:

- if $v \in V$, then $\{v\} \in K$,
- if $\tau \in K$ and $\tau' \subset \tau$ is a nonempty subset of V, then $\tau' \in K$.

The dimension of a simplex τ is the cardinality of τ minus 1. We denote the dimension of a simplex τ by $\dim \tau$.

A simplex τ is said to be a q-simplex if $\dim \tau = q$. The dimension of complex K is defined as the supremum of the dimension of the simplices of K.

A subcomplex of an abstract simplicial complex K is said to be a full subcomplex, denoted L if whenever a set of vertices of L is a simplex in K.

The complex of curves on an orientable surface S is the abstract simplicial complex such that the vertices are the isotopy classes of nontrivial circles. It is denoted by $C(\Sigma)$ and introduced by Harvey [5]. Similarly, the complex of curves $C(N)$ on a nonorientable surface N is the abstract simplicial complex such that the vertices are the isotopy classes of nontrivial circles. In this complex of curves, we take one-sided vertices as well as two-sided vertices.

A set of vertices $\{v_0, v_1, ..., v_q\}$ forms a $q - simplex$ if and only if $i(v_j, v_k) = 0$ for all $0 \leq j, k \leq q$. In particular, two distinct vertices α_1 is connected to α_2 in the complex of curves by an edge if and only if $i(\alpha_1, \alpha_2) = 0$.

Let Σ be a connected orientable surface of genus g with n boundary components such that $2g + n \geq 4$. The dimension of the complex of curves $C(\Sigma)$ is $3g + n - 4$.

If we take only the set of isotopy classes of nonseparating circles on Σ as vertices, then the complex of nonseparating circles denoted by $Nonsep(\Sigma)$ is defined as follows:

Let **A** be the set of isotopy classes of nonseparating circles on Σ. Let $Nonsep(\Sigma)$ denote the complex of nonseparating curves. The complex $Nonsep(\Sigma)$ is the subcomplex of $C(\Sigma)$ with the set **A** such that $\{\gamma_0, \gamma_1, \gamma_2, ..., \gamma_q\}$ forms a q-simplex if and only if it is a q-simplex in the complex $C(\Sigma)$.

Let α and β be the isotopy classes of nonseparating circles on Σ. Using the set **A**, Schmutz Schaller defined a graph $G(\Sigma)$, α is connected to β by an edge if and only if $i(\alpha,\beta) = 1$ ([17]). The graph $G(\Sigma)$ is defined for spheres as well. We also note that the graph $G(\Sigma)$ is the one skeleton of the complex $HT(\Sigma)$.

2.8 The Mapping Class Group of a Surface

Let Σ be a connected orientable surface of genus g with n boundary components. The mapping class group of Σ, denoted by $Mod(\Sigma)$ is the group of isotopy classes of orientation-preserving homeomorphism of Σ where isotopies fix the boundary pointwise. If we consider the group of isotopy classes of all self homeomorphisms of Σ, we call the extended mapping class group denoted by $Mod^*(\Sigma)$. The mapping class group $Mod(\Sigma)$ is the subgroup of index two in $Mod^*(\Sigma)$.

If N denotes a connected nonorientable surface of genus g, the mapping class group $Mod(N)$ of N is group of isotopy classes of all homeomorphisms of N .

Example 2.8.1 *The mapping class group of the closed disk is trivial (see, pg 47 [4]).*

Example 2.8.2 *The mapping class group of the annulus is isomorphic to $\mathbb{Z}$ (see, pg 51 [4]).*

Example 2.8.3 *The mapping class group of the torus is isomorphic to $SL(2, \mathbb{Z})$ (see, pg 52 [4]).*

Example 2.8.4 *The mapping class group of the real projective plane is trivial and the mapping class group of the real projective plane with one puncture (marked point) is isomorphic to $\mathbb{Z}_2$ (see, [13]).*

2.9 Results about Automorphisms of Various Complexes on Curves

Let Σ be a connected orientable surface of genus g with n holes. Let $H : \Sigma \to \Sigma$ be a homeomorphism of Σ and c be a nontrivial simple closed curve. Then, $H(c)$ is also nontrivial circle. If H is isotopic to F and c is isotopic to d, then $H(c)$ is isotopic to $F(d)$. Therefore, the group $Mod^*(\Sigma)$ of Σ acts on the complex of curves $C(\Sigma)$ on Σ as simplicial automorphisms. In other words, there is a natural group homomorphism

$$Mod^*(\Sigma) \to AutC(\Sigma).$$

We have the following result ([11], [12] and [14]).

Theorem 2.9.1 *Let Σ be a compact, connected, orientable surface of genus g with n holes. Suppose that Σ is neither a sphere with at most four holes, nor a torus with at most two holes, nor a closed surface of genus two. Then the natural map $Mod^*(\Sigma) \longrightarrow AutC(\Sigma)$ is an isomorphism. If Σ is a closed surface of genus two then this map is onto and the kernel is the subgroup of order two generated by the hyperelliptic involution.*

The following theorem extends this result to nonorientable surfaces ([1]).

Theorem 2.9.2 *Let N be a compact, connected, nonorientable surface of genus g with n holes. Suppose that $g + n \geq 5$. Then the natural map $Mod(N) \to AutC(N)$ is an isomorphism.*

For the complex of nonseparating circles $Nonsep(\Sigma)$ of an orientable surface Σ of $g \geq 2$, the automorphism group of $Nonsep(\Sigma)$ is isomorphic to the group $Mod^*(\Sigma)$ of Σ by E. Irmak [9].

Theorem 2.9.3 *Suppose that $g \geq 2$. If Σ is a closed orientable surface of genus 2, then $AutNonsep(\Sigma)$ is isomorphic to the extended mapping class group $Mod^*(\Sigma)$ modulo its center. If Σ is not a closed orientable surface of genus 2, then $AutNonsep(\Sigma)$ is isomorphic to the extended mapping class group $Mod^*(\Sigma)$.*

Theorem 2.9.4 *Let Σ be a compact, connected, orientable surface of genus $g \geq 1$. Then $AutG(\Sigma)$ is isomorphic to the group $Mod^*(\Sigma)$ modulo the center.*

CHAPTER 3

Automorphisms of the Hatcher-Thurston complex

Let Σ denote a compact, connected, orientable surface of genus $g \geq 1$ with $n \geq 0$ holes. In this chapter, we consider the automorphism group of the Hatcher-Thurston complex for orientable surfaces.

Definition 3.0.5 *Let ϵ be an embedded arc on a surface Σ. If the boundary of ϵ is a subset of the boundary of Σ and ϵ is transversal to the boundary of Σ, then ϵ is said to be properly embedded on the surface Σ. If it cannot be deformed into the boundary Σ so that the endpoints of ϵ remain in the boundary Σ during the deformation, then we say that ϵ is nontrivial.*

Definition 3.0.6 *For a nonseparating circle c on an orientable surface Σ, a simplicial graph X_c is defined as follows:*

- *the vertices are isotopy classes of nonseparating circles such that they are dual to c on the surface Σ.*

Two vertices α and β are connected by an edge if and only if α and β are dual; that is $i(\alpha, \beta) = 1$.

Lemma 3.0.7 *Let Σ be a connected orientable surface of genus $g \geq 1$ and c be a nonseparating circle on Σ. Then the graph X_c is connected.*

Proof. Let δ and δ' denote two distinct vertices in the graph X_c. We need to find a path from δ to δ' in X_c. Let the representatives of δ and δ' be d and d', respectively;

such that they have mininal intersection. Obviously, $|d \cap c| = 1$ and $|d' \cap c| = 1$. We may also suppose that d and d' intersect c at different points. To prove this lemma we use induction on $|d \cap d'|$.

• Suppose that d and d' are disjoint. Then, δ is connected to $t_c(\delta)$ by an edge and also, $t_c(\delta)$ is connected to δ' by an edge in X_c. We note that t_c is the Dehn twist about the isotopy class of c.

• Suppose that d is dual to d'. Then, δ and δ' are connected by an edge by definition.

• Let the representatives of any two vertices intersect less than k times. Suppose that these vertices are joined by a path in the graph X_c.

• Suppose that $|d \cap d'| = k > 1$. We denote a regular neighborhood of c by U so that the intersection of $d \cup d'$ and U is a pair of disjoint arcs. We denote the complement of the interior of U in Σ by M. If we denote the part of d and d' on M by ϵ and ϵ', respectively; so that they are nontrivial properly embedded arcs. Let ϵ and ϵ' be oriented so that they both start on ∂_1 and finish on ∂_2, where ∂_1 and ∂_2 are the boundary components of U. Then one define an arc as follows:

We start on ∂_1 of M, on one side of the initial point of ϵ' and follow along ϵ' without intersecting ϵ', till the final intersection point of ϵ and ϵ' along ϵ. We continue ϵ, without intersecting $\epsilon \cup \epsilon'$, up to arrive ∂_2. On the correct side of ϵ', we perform this. Otherwise, we modify our initial side from the starting and track the construction. Then we have an arc, say ϵ''. Clearly, ϵ'' is a nontivial properly embedded arc on M because ∂_1 and ∂_2 are connected by ϵ''. Moreover, $|\epsilon \cap \epsilon''| < k$, because we eliminated at least one intersection point with ϵ. Also, $|\epsilon'' \cap \epsilon'| = 0$, because we do not intersect ϵ'.

The endpoints of ϵ'' stay in distinct components of $U \setminus (d \cup d')$. These endpoints of ϵ'' are joined by an arc in U disjoint from $U \cap d$ and intersection each of c and $U \cap d'$ only one point. It follows that a circle d'' intersects c at one point, and so, the isotopy class δ'' of d'' is a vertex in X_c. Also, $|d'' \cap d'| = 1$, since $|\epsilon'' \cap \epsilon'| = 0$. Hence, δ' is conncected to δ'' by an edge in the graph X_c. Moreover, we have $|d'' \cap d| = |\epsilon'' \cap \epsilon| < k$. By induction assumption, there is a path in X_c such that δ'' is connected δ by this path in the graph X_c. The lemma follows. $\square$

3.1 Action of automorphisms of the Hatcher-Thurston complex $HT(\Sigma)$

In this subsection, an action of the automorphism group $AutHT(\Sigma)$ of the Hatcher-Thurston complex $HT(\Sigma)$ is defined on the set of nonseparating circles on an orientable surface Σ.

Let $f : HT(\Sigma) \to HT(\Sigma)$ denote an automorphism of the Hatcher-Thurston complex $HT(\Sigma)$ of Σ. Let c denote a nonseparating circle on Σ and γ denote the isotopy class of c. For the isotopy class γ, we pick pairwise disjoint nonseparating circles $c_2, c_3, ..., c_g$ on Σ such that $u = \left\langle \gamma, \gamma_2, ..., \gamma_g \right\rangle$ is a cut system for Σ. If we pick another nonseparating circle d on Σ such that it is dual to c and does not intersect any of c_i, then we have another cut system $v = \left\langle \delta, \gamma_2, ..., \gamma_g \right\rangle$. The vertex u is connected to the vertex v by an edge in the graph $HT(\Sigma)$. Then, the vertex $f(u)$ is also connected to $f(v)$ by an edge, because f is an automorphism of $HT(\Sigma)$. Hence, the set difference $f(u) - f(v)$ has only one isotopy class of nonseparating circle. Now, we define $\tilde{f}(\gamma)$ to be this unique class.

We note that the vertex u has only one element γ, provided that $g = 1$. Therefore, $\tilde{f}(\gamma)$ is the unique class in $f\langle\gamma\rangle$, so that $\left\langle \tilde{f}(\gamma) \right\rangle = f\langle\gamma\rangle$.

Now, we need to show that this action is well-defined.

Lemma 3.1.1 *The definition of $\tilde{f}(\gamma)$ is independent of the choice of the nonseparating circle d, for a fixed set of nonseparating circles* $\{c_2, c_3, ..., c_g\}$.

Proof. Let $v \leftrightarrow w \leftrightarrow u \leftrightarrow v$ be a triangle in $HT(\Sigma)$. Hence, we have $v - w = v - u$.

Let κ be an isotopy class of a nonseparating circle k. For the isotopy class κ such that $\left\langle \kappa, \gamma_2, ..., \gamma_g \right\rangle$ is a cut system for the surface Σ. Let the cut system $\left\langle \kappa, \gamma_2, ..., \gamma_g \right\rangle = \langle\kappa\rangle$.

Let d' be a nonseparating circle such that $|d' \cap c| = 1$, $|d' \cap d| = 1$, and let d' be disjoint from c_i for $i \geq 2$. Let δ' and γ_i denote the isotopy classes of d' and c_i for $i \geq 2$. Then $\langle\delta'\rangle = \left\langle \delta', \gamma_2, ..., \gamma_g \right\rangle$ is also a cut system. Moreover, $\langle\gamma\rangle \leftrightarrow \langle\delta\rangle \leftrightarrow \langle\delta'\rangle \leftrightarrow \langle\gamma\rangle$ is a triangle in $HT(\Sigma)$. Therefore, $f\langle\gamma\rangle \leftrightarrow f\langle\delta\rangle \leftrightarrow f\langle\delta'\rangle \leftrightarrow f\langle\gamma\rangle$ is a triangle in $HT(\Sigma)$, because f is an automorphism. Then, we have $f\langle\gamma\rangle - f\langle\delta\rangle = f\langle\gamma\rangle - f\langle\delta'\rangle$.

Now assume that d' is any nonseparating circle such that $|d' \cap c| = 1$ and it is disjoint from c_i for $i \geq 2$. So, we have two vertices δ and δ' in the graph X_c. It follows from connectivity of the graph X_c that we can find a sequence $\delta = \delta_1, \delta_2, ..., \delta_n = \delta'$ of vertices in X_c such that δ_i and δ_{i+1} are connected by an edge for all $i = 1, 2, ..., n-1$. Then, we obtain that $f\langle\gamma\rangle - f\langle\delta_i\rangle = f\langle\gamma\rangle - f\langle\delta_{i+1}\rangle$. Hence, $f\langle\gamma\rangle - f\langle\delta\rangle = f\langle\gamma\rangle - f\langle\delta'\rangle$. The proof completes. □

Lemma 3.1.2 *The definition of $\tilde{f}(\gamma)$ is independent of all choices for the nonseparating circle c.*

Proof.

Assume that $\left\{d, c_2, c_3, ..., c_g\right\}$ and $\left\{d', c'_2, c'_3, ..., c'_g\right\}$ are two choices in the definition of $\tilde{f}(\gamma)$. We need to show that both options bring about the same result. If $u = \left\langle\gamma, \gamma_2, ..., \gamma_g\right\rangle$, $v = \left\langle\delta, \gamma_2, ..., \gamma_g\right\rangle$, $u' = \left\langle\gamma, \gamma'_2, ..., \gamma'_g\right\rangle$ and $v' = \left\langle\delta', \gamma'_2, ..., \gamma'_g\right\rangle$ such that $u \leftrightarrow v$ and $u' \leftrightarrow v'$. Now, we need to prove that $f(u') - f(v') = f(u) - f(v)$.

If the genus g of the surface Σ is equal to 1, we don't have any nonsepataing circles c_i and c'_i and by Lemma 3.1.1, the conclusion of the lemma follows.

Suppose that $g \geq 2$. Firstly, assume that u' and u are joined by an edge. So, there are elements γ_{i_0} and γ'_{j_0} in u and u', respectively; such that their representatives c_{i_0} and c'_{j_0} of γ_{i_0} and γ'_{j_0} intersect transversely once and that $u \setminus \{\gamma_{i_0}\} = u' \setminus \{\gamma'_{j_0}\}$. If necessary, after reindexing, one may suppose that $\gamma_{i_0} = \gamma_2$ and $\gamma'_{j_0} = \gamma'_2$, so that $\gamma'_i = \gamma_i$ for $i \geq 3$. Let e denote a nonseparating circle, $|e \cap c| = 1$, $|e \cap c_i| = 0$ and $|e \cap c'_i| = 0$ for $i \geq 2$. Let $s = \left\langle\epsilon, \gamma_2, \gamma_3, ..., \gamma_g\right\rangle$ and $s' = \left\langle\epsilon, \gamma'_2, \gamma'_3, \gamma'_4, ..., \gamma'_g\right\rangle = \left\langle\epsilon, \gamma'_2, \gamma_3, \gamma_4, ..., \gamma_g\right\rangle$. Since $u \leftrightarrow u' \leftrightarrow s' \leftrightarrow s \leftrightarrow u$ constitute a rectangle in $HT(\Sigma)$ and also, $f(u) \leftrightarrow f(u') \leftrightarrow f(s') \leftrightarrow f(s) \leftrightarrow f(u)$ is a rectangle in $HT(\Sigma)$, because f is an automorphism. Therefore, we have $f(u) - f(s) = f(u') - f(s')$. By Lemma 3.1.1, $f(u) - f(s) = f(u) - f(v)$ and $f(u') - f(s') = f(u') - f(v')$. It follows that $f(u) - f(v) = f(u') - f(v')$.

Now, we regard the general case. If Q denotes the surface obtained by cutting Σ along the nonseparating circle c, then Q is an orientable surface of genus $g > 0$ because

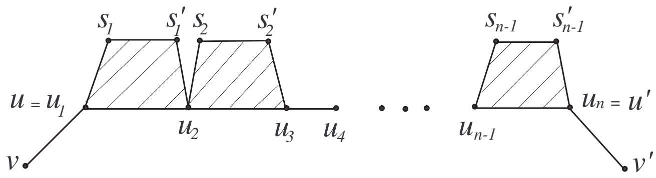

Figure 3.1: A path in $HT(\Sigma)$

$g \geq 2$. As $|c \cap c_i| = 0$ and $|c \cap c_i'| = 0$ for $i \geq 2$, we may take them as nonsepataing circles on Q. Then $r = \langle \gamma_2, \gamma_3, ..., \gamma_g \rangle$ and $r' = \langle \gamma_2', \gamma_3', ..., \gamma_g' \rangle$ are two cut systems on Q. The complex $HT(Q)$ of the surface Q is connected by Theorem 2.6.1. By the connectivity of $HT(Q)$, we can find a sequence $r = r_1, r_2, r_3, ..., r_n = r'$ of cut systems on the surface Q such that r_i and r_{i+1} are joined by an edge. If u_i denotes the cut system on Σ obtained from r_i by adding γ, then we obtain a path $u = u_1, u_2, ..., u_n = u'$ in $HT(\Sigma)$ of the surface Σ. For every $i = 1, 2, ..., g-1$, pick vertices s_i and s_i' as above so that $u_i \leftrightarrow u_{i+1} \leftrightarrow s_i' \leftrightarrow s_i \leftrightarrow u_i$ is a rectangle. We proved that $f(u_i) - f(s_i) = f(u_{i+1}) - f(s_i')$. Also, using Lemma 3.1.1, we get $f(u_{i+1}) - f(s_i') = f(u_{i+1}) - f(s_{i+1})$. Then, $f(u_1) - f(s_1) = f(u_n) - f(s_{n-1}')$. By Lemma 3.1.1, we obtain that $f(u) - f(v) = f(u') - f(v')$. This finishes the proof of the lemma. □

Lemma 3.1.3 *Let c and d be any two nonseparating circles. Let γ and δ denote the isotopy classes of c and d, respectively. If γ is dual to δ, $\tilde{f}(\gamma)$ is dual to $\tilde{f}(\delta)$.*

Proof. We consider two cut systems u and v consisting of γ and δ, respectively. Then, we may find nonseparating circles $c_2, c_3, ..., c_g$ on the surface Σ such that $u = \langle \gamma, \gamma_2, ..., \gamma_g \rangle$ and $v = \langle \delta, \gamma_2, ..., \gamma_g \rangle$ are two vertices in $HT(\Sigma)$. Here, γ_i is the isotopy class of c_i for $i = 2, \cdots, g$. Since $i(\gamma, \delta) = 1$, u is connected to v by an edge in $HT(\Sigma)$. $f(u)$ is also connected to $f(v)$ by an edge in the complex $HT(\Sigma)$, because f is an element of automorphism group of the complex $HT(\Sigma)$. It follows from the definition of $\tilde{f}$ that $\{\tilde{f}(\gamma)\} = f(u) - f(v)$ and $\{\tilde{f}(\delta)\} = f(v) - f(u)$. As $f(u)$ is connected to $f(v)$ by an edge, we get $i(\tilde{f}(\gamma), \tilde{f}(\delta)) = 1$. □

Lemma 3.1.4 *Let γ be the isotopy class of a nonnseparating circle c. Let f and g be two automorphisms of $HT(\Sigma)$. $\tilde{fg}(\gamma) = \tilde{f}(\tilde{g}(\gamma))$.*

Proof. Let $\gamma_2, \gamma_3, ..., \gamma_g$ and δ be vertices in the graph $G(\Sigma)$ (see Chapter 2) such that $u = \langle \gamma, \gamma_2, ..., \gamma_g \rangle$ and $v = \langle \delta, \gamma_2, ..., \gamma_g \rangle$ are different vertices in $HT(\Sigma)$ which are joined by an edge. By the definition of $\tilde{g}$, we have $\{\tilde{g}(\gamma)\} = g(u) - g(v)$. Also, since f and g are two elements of the group of automorphisms of the complex $HT(\Sigma)$, we have $\{\widetilde{fg}(\gamma)\} = fg(u) - fg(v)$. Because $g(u)$ is connected by an edge to $g(v)$ in the complex $HT(\Sigma)$, using these vertices, we define $\tilde{f}(\tilde{g}(\gamma))$ as follows:

$$\{\tilde{f}(\tilde{g}(\gamma))\} = f(g(u)) - f(g(v)) = (fg)(u) - (fg)(v) = \{(\widetilde{fg})(\gamma)\}.$$

The proof of the lemma is completed. □

Proposition 3.1.5 *The mapping $\tilde{f}$ is an automorphism of $G(\Sigma)$.*

Proof. Since $\tilde{f}(\gamma)$ is well-defined for a vertex γ in the graph $G(\Sigma)$, we have a well-defined map $\tilde{f} : G(\Sigma) \to G(\Sigma)$. If we take two vertices γ and δ connected by an edge in the graph $G(\Sigma)$, then $i(\gamma, \delta) = 1$ by the definition of the graph $G(\Sigma)$ (see Chapter 2). By Lemma 3.1.3, $i(\tilde{f}(\gamma), \tilde{f}(\delta)) = 1$. Hence, $\tilde{f}$ is a simplicial map.

Let g be the inverse automorphism of f. Then $\tilde{f}\tilde{g}$ and $\tilde{g}\tilde{f}$ are both the identity automorphisms. Because if I denotes the identity automorphism, then $\tilde{I}(\gamma) = \gamma$ for all vertices in the graph $G(\Sigma)$. It follows that $\tilde{f} : G(\Sigma) \to G(\Sigma)$ is a bijection map. Hence, $\tilde{f}$ is an automorphism of the graph $G(\Sigma)$. □

In the following proposition, we say that $\tilde{f}$ is also an automorphism of the nonseparating curve complex $Nonsep(\Sigma)$ (see Chapter 2) for the closed orientable surfaces.

Proposition 3.1.6 *Let Σ denote a closed orientable surface of genus $g \geq 2$. $\tilde{f} : Nonsep(\Sigma) \to Nonsep(\Sigma)$ is an automorphism.*

Proof. By Proposition 3.1.5, $\tilde{f}$ is a bijection, because the vertices of the complex $Nonsep(\Sigma)$ are the same as the set of vertices of the graph $G(\Sigma)$ on the surface Σ. Therefore, it is enough to prove that $\tilde{f}$ is a simplicial map on the complex $Nonsep(\Sigma)$. Let α and β be two disinct vertices of $Nonsep(\Sigma)$, having disjoint representatives a and b on Σ, respectively. We regard the following two cases:

Case (i): Suppose that the surface $\Sigma_{a \cup b}$ is connected. Then we can complete the set

$\{\alpha, \beta\}$ to vertex u in $HT(\Sigma)$. Since f is an automorphism of the complex $HT(\Sigma)$, $f(u)$ is also a vertex in the complex $HT(\Sigma)$ and moreover, $\tilde{f}(\alpha)$ and $\tilde{f}(\beta)$ are in $f(u)$. So, $\tilde{f}(\alpha)$ and $\tilde{f}(\beta)$ have disjoint representatives on the surface Σ.

Case (ii): Suppose that the surface $\Sigma_{a \cup b}$ is not connected. a and b can be completed to a circle configuration as shown in Figure 3.2. Let us take a maximal chain $\{c_1, ..., c_{2g+1}\}$ with $i(\gamma_i, \gamma_{i+1}) = 1$ and $i(\gamma_i, \gamma_j) = 0$ for $|i - j| > 1$, where γ_i is the isotopy class of c_i as shown in the Figure 3.2 for $g = 5$ case (we can take similar maximal chains for the other cases). Note that the surface $\Sigma_{c_i \cup c_j}$ is connected for any i, j becuase c_i and c_j are nonseparating circles. If $i(\gamma_i, \gamma_j) = 0$, $i(\tilde{f}(\gamma_i), \tilde{f}(\gamma_j)) = 0$ by Case (i). Now, using Lemma 3.1.3, if $i(\gamma_i, \gamma_j) = 1$, we have $i(\tilde{f}(\gamma_i), \tilde{f}(\gamma_j)) = 1$. Thus $\{\tilde{f}(\gamma_1), ..., \tilde{f}(\gamma_{2g+1})\}$ is a maximal chain on the surface Σ.

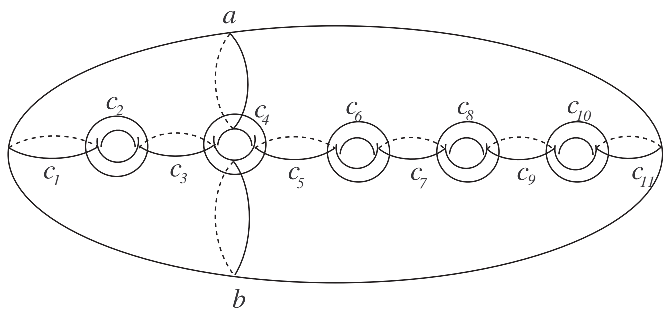

Figure 3.2: a, b and a chain

Since a is the nonseparating circle, we also get $\Sigma_{a \cup c_i}$ is a connected surface for any i. If α and γ_i are disjoint, $\tilde{f}(\alpha)$ and $\tilde{f}(\gamma_i)$ are disjoint by Case (i). If α is dual to γ_i, $\tilde{f}(\alpha)$ is dual to $\tilde{f}(\gamma_i)$ by Lemma 3.1.3. Similarly, we say this argument for the nonseparating circle b. If β and γ_i are disjoint, $\tilde{f}(\beta)$ and $\tilde{f}(\gamma_i)$ are disjoint by Case (i). Using Lemma 3.1.3, if β is dual to γ_i, then $\tilde{f}(\beta)$ is dual to $\tilde{f}(\gamma_i)$.

We notice that $i(\alpha, \gamma_{2k}) = i(\beta, \gamma_{2k}) = 1$ for some $k \in \{2, 3, ..., g-1\}$ and α and β are disjoint from any other γ_i. Therefore, $i(\tilde{f}(\alpha), \tilde{f}(\gamma_{2k})) = i(\tilde{f}(\beta), \tilde{f}(\gamma_{2k})) = 1$ and $\tilde{f}(\alpha)$ and $\tilde{f}(\beta)$ are disjoint from any other $\tilde{f}(\gamma_i)$.

Let c'_i, a' and b' be representatives of $\tilde{f}(\gamma_i)$, $\tilde{f}(\alpha)$ and $\tilde{f}(\beta)$, respectively; such that the circles c'_i, a' and b' intersect minimally with each other for every i. By Lemma 3.1.3, since $|a \cap c_{2k}| = 1$ and $|b \cap c_{2k}| = 1$, $|a' \cap c'_{2k}| = 1$ and $|b' \cap c'_{2k}| = 1$. As circles a' and b'

are disjoint from circles in the two chains $c'_1 \cup c'_2 \cup \cdots \cup c'_{2k-1}$ and $c'_{2k+1} \cup c'_{2k+2} \cup \cdots \cup c'_{2g+1}$, and also because the complement of these chains is the union of two annuli, the circles a' and b' have to be disjoint. Since we have only two circles on the disjoint union of two annuli up to isotopy and these circles are disjoint. Hence, $\tilde{f}(\alpha)$ is disjoint from $\tilde{f}(\beta))$. It follows that $\tilde{f}$ is a simplical map on the complex $Nonsep(\Sigma)$. Since $\tilde{f}$ is bijection, it is an automorphism of $Nonsep(\Sigma)$. □

Remark 3.1.7 *If Σ is a closed orientable surface of genus $g \geq 2$, using Proposition 3.1.6 and the result in [9], it follows that f is induced by a surface homeomorphism of the surface Σ.*

3.2 Automorphisms of $HT(\Sigma)$

In this section, the main result is proved for compact orientable surfaces of genus $g \geq 1$ and then we give its corollary. Finally, we give another proof of this result for closed orientable surfaces of genus $g \geq 2$.

Theorem 3.2.1 *Let Σ be a compact, connected, orientable surface of genus $g \geq 1$ and f be an automorphism of $HT(\Sigma)$. Then $\Phi : AutHT(\Sigma) \to AutG(\Sigma)$ given by $f \mapsto \tilde{f}$ is an isomorphism.*

Proof. Using the results of the previous subsection, $\Phi(f) = \tilde{f}$ is a well-defined automorphism of $AutG(\Sigma)$. Φ is a group homomorphism:

$$\Phi(fg) = (\tilde{fg}).$$

On the other hand, $(\tilde{fg}) = \tilde{f}\tilde{g}$ by Lemma 3.1.4. Therefore,

$$\Phi(fg) = (\tilde{fg}) = \tilde{f}\tilde{g} = \Phi(f)\Phi(g).$$

For an element f in $AutHT(\Sigma)$, if $\tilde{f}$ is the identity automorphism of $G(\Sigma)$, then $f(\left\langle \gamma_1, \gamma_2, ..., \gamma_g \right\rangle) = \left\langle \tilde{f}(\gamma_1), \tilde{f}(\gamma_2), ..., \tilde{f}(\gamma_g) \right\rangle$ for any isotopy classes $\gamma_1, \gamma_2, ..., \gamma_g$ of nonseparating circles. It follows that f acts trivially on the complex $HT(\Sigma)$. Therefore, Φ is injective.

If g is an element of the group of automorphisms of $G(\Sigma)$, g is induced by a homeomorphism F of Σ using Theorem 2.9.4. Also, F induces an automorphism f of $HT(\Sigma)$ and $\tilde{f} = g$. Hence, Φ is an isomorphism. The proof of the theorem is completed. □

Corollary 3.2.2 *Let Σ denote a compact, connected, orientable surface of genus $g \geq 1$ with $n \geq 0$ holes. Suppose that $(g, n) \neq (1, 0), (1, 1), (1, 2), (2, 0)$. Then, $AutHT(\Sigma)$ is isomorphic to $Mod^*(\Sigma)$. If (g, n) is one of $(1, 0), (1, 1), (1, 2), (2, 0)$, then $AutHT(\Sigma)$ is isomorphic to $Mod^*(\Sigma)/\mathbb{Z}_2$. In other words, $AutHT(\Sigma)$ is isomorphic to $Mod^*(\Sigma)$ modulo its center for all cases.*

Proof. The proof of the corollary follows from Theorem 3.2.1 and Theorem 2.9.4. □

Theorem 3.2.3 *Let Σ denote a closed orientable surface of genus $g \geq 2$ and f be an automorphism of the complex $HT(\Sigma)$. f is induced by a homeomorphism of the surface Σ.*

Proof. The proof follows from Proposition 3.1.6 and Theorem 3.1.7. □

CHAPTER 4

The Hatcher-Thurston complex for nonorientable surfaces

Let N be a compact, connected nonorientable surface of genus $g \geq 1$ with $n \geq 0$ holes. We note that the genus of a nonorientable surface is the maximum number of projective planes in a connected sum decomposition. We will define cut systems and the Hatcher-Thurston complex for a nonorientable surface N in this chapter.

Definition 4.0.4 *Let a be a one-sided circle. If either $g = 1$ or $g \geq 2$ and the surface N_a is nonorientable, we say that a is essential. A characteristic one-sided circle is a one-sided circle which is not essential.*

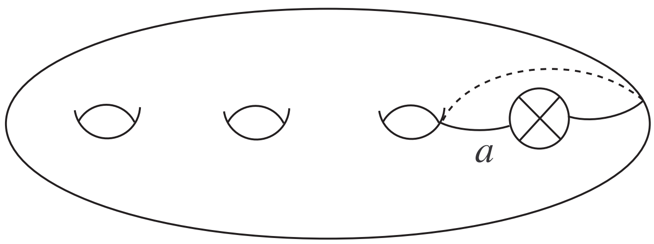

Figure 4.1: a is an essential one-sided circle.

4.1 Cut Systems

Let a set $\{a_1, a_2, ..., a_g\}$ be pairwise disjoint essential one-sided circles on N. Let the set of their isotopy classes denote by $\langle \alpha_1, \alpha_2, ..., \alpha_g \rangle$. Let us cut the surface N along all a_i. Then if the resulting surface is connected, and so it is a sphere with g holes,

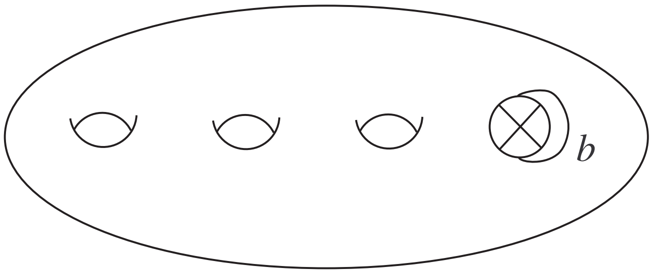

Figure 4.2: b is a characteristic one-sided circle.

we say that the set $\{a_1, a_2, ..., a_g\}$ is a geometric cut system on N. If $\{a_1, a_2, ..., a_g\}$ is a geometric cut system, then we say that the set of the isotopy classes $\left\langle \alpha_1, \alpha_2, ..., \alpha_g \right\rangle$ is a cut system.

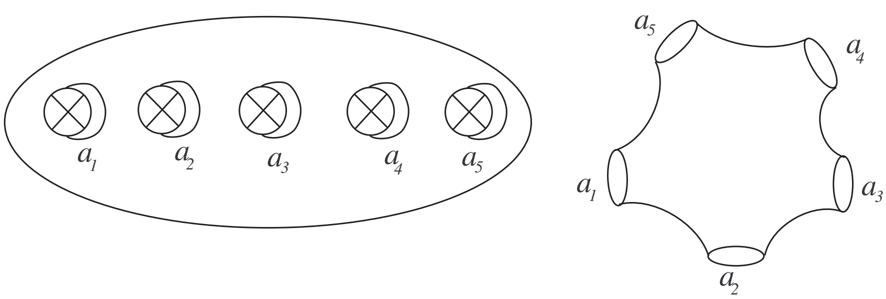

Figure 4.3: $\{a_1, a_2, a_3, a_4, a_5\}$ is a geometric cut system on a closed nonrientable surface of genus 5.

4.2 The Elementary Move

Let $\{a_1, a_2, ..., a_g\}$ be a geometric cut system on N. Assume that a' be an essential one-sided circle on N disjoint from a_j with $j \neq i$, $1 \leq i \leq g$, and dual to a_i. If we replace a_i by a' in the set $\{a_1, a_2, ..., a_g\}$, we obtain another cut system. An elementary move is the operation of replacing the geometric cut system $\{a_1, ..., a_i, ..., a_g\}$ by the geometric cut system $\{a_1, ..., a', ..., a_g\}$, and also the corresponding operation of replacing the cut system $\left\langle \alpha_1, ..., \alpha_i, ..., \alpha_g \right\rangle$ by the cut system $\left\langle \alpha_1, ..., \alpha', ..., \alpha_g \right\rangle$. We will usually throw away the unchanged circles and we will record $\langle \alpha_i \rangle \leftrightarrow \langle \alpha' \rangle$.

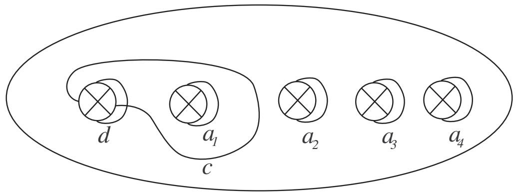

Figure 4.4: $\langle d, a_1, a_2, a_3, a_4 \rangle \leftrightarrow \langle c, a_1, a_2, a_3, a_4 \rangle$

4.3 The Hatcher-Thurston Graph

If we take cut systems on N as the set of vertices, we obtain the Hatcher-Thurston graph of N. If one vertex is obtained from another by an elementary move, then these two vertices are connected by an (unordered) edge corresponding to this move. At this moment we have already the graph consisting of just described vertices and edges. It is denoted by $HT^1(N)$.

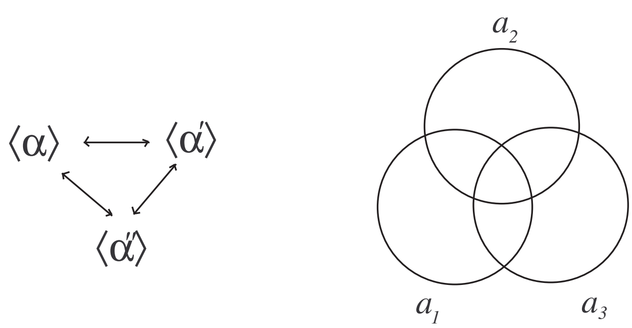

Figure 4.5: A triangle in the Hatcher-Thurston graph for a nonrientable surface.

Let $(\alpha_1, ..., \alpha_n)$ be a sequence of cut systems. If each consecutive pair in $(\alpha_1, ..., \alpha_n)$ is connected by an edge, $(\alpha_1, ..., \alpha_n)$ forms a path in $HT^1(N)$. If $\alpha_1 = \alpha_n$, then we say that the path is closed. Also again, we have three types of special closed paths in the Hatcher-Thurston graph $HT^1(N)$ of N.

1. **Triangles.** Let three vertices have $g - 1$ common isotopy classes. If the remain-

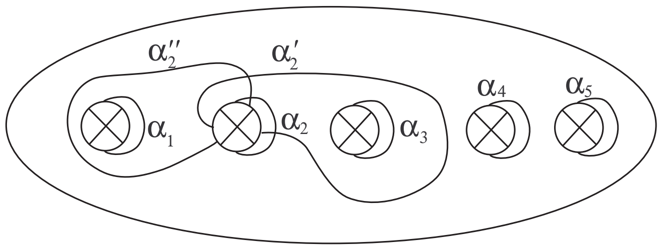

Figure 4.6: This figure is an example for the triangle in the Hatcher-Thurston graph on a closed nonorientable surface of genus 5, $\langle \alpha_2 \rangle \longleftrightarrow \left\langle \alpha_2' \right\rangle \longleftrightarrow \left\langle \alpha_2'' \right\rangle \longleftrightarrow \langle \alpha_2 \rangle$.

ing classes α_i, α_i', α_i'' are dual pairwisely, then this closed path is a triangle. This triangle is denoted by $\langle \alpha_i \rangle \longleftrightarrow \left\langle \alpha_i' \right\rangle \longleftrightarrow \left\langle \alpha_i'' \right\rangle \longleftrightarrow \langle \alpha_i \rangle$. In other words, let $\left\langle \alpha_1, ..., \alpha_i, ..., \alpha_g \right\rangle = \langle \alpha_i \rangle$, $\left\langle \alpha_1, ..., \alpha_i^{'}, ..., \alpha_g \right\rangle = \left\langle \alpha_i^{'} \right\rangle$ and $\left\langle \alpha_1, ..., \alpha_i^{''}, ..., \alpha_g \right\rangle = \left\langle \alpha_i^{''} \right\rangle$. If α_i, α_i', α_i'' satisfy $i(\alpha_i, \alpha_i') = i(\alpha_i, \alpha_i'') = i(\alpha_i', \alpha_i'') = 1$, then we have a triangle. (see Figure 4.6.)

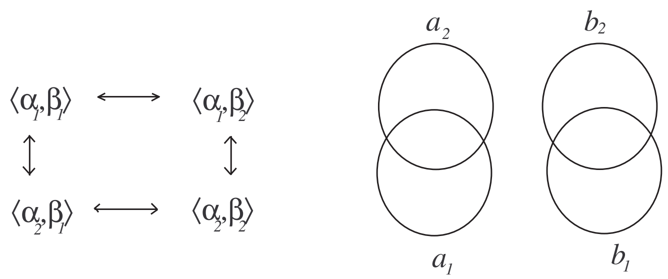

Figure 4.7: A rectangle in the Hatcher-Thurston graph for a nonrientable surface.

2. **Rectangles.** Let four vertices have $g - 2$ common isotopy classes. If the other classes α_1, α_2, β_1, β_2 have representatives a_1, a_2, b_1, b_2, respectively; as in Figure 4.7, then this closed path is a rectangle. More expilicitly, $\left\langle \alpha_1, \beta_1, \delta_1 ..., \delta_{g-2} \right\rangle = \langle \alpha_1, \beta_1 \rangle$, $\left\langle \alpha_1, \beta_2, \delta_1 ..., \delta_{g-2} \right\rangle = \langle \alpha_1, \beta_2 \rangle$, $\left\langle \alpha_2, \beta_1, \delta_1 ..., \delta_{g-2} \right\rangle = \langle \alpha_2, \beta_1 \rangle$ and $\left\langle \alpha_2, \beta_2, \delta_1 ..., \delta_{g-2} \right\rangle = \langle \alpha_2, \beta_2 \rangle$ as in Figure 4.7, then the corresponding rectangle is denoted by $\langle \alpha_1, \delta_1 \rangle \longleftrightarrow \langle \alpha_1, \beta_2 \rangle \longleftrightarrow \langle \alpha_2, \beta_2 \rangle \longleftrightarrow \langle \alpha_2, \beta_1 \rangle \longleftrightarrow \langle \alpha_1, \beta_1 \rangle$.

3. **Pentagons.** Let five vertices have $g - 2$ common isotopy classes. If the other classes α_1, α_2, α_3, α_4, α_5 have representatives a_1, a_2, a_3, a_4, a_5, respectively; as in Figure 4.8, then this closed path is a pentagon. In other words,

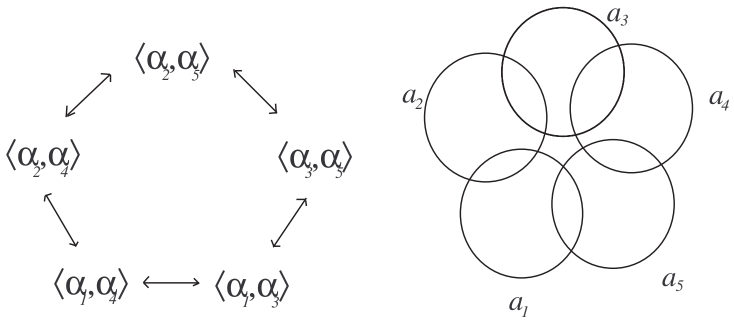

Figure 4.8: A pentagon in the Hatcher-Thurston graph for a nonrientable surface.

$\left\langle\alpha_1,\alpha_4,\delta_1,..,\delta_{g-2}\right\rangle = \langle\alpha_1,\alpha_4\rangle$, $\left\langle\alpha_2,\alpha_4,\delta_1,..,\delta_{g-2}\right\rangle = \langle\alpha_2,\alpha_4\rangle$, $\left\langle\alpha_2,\alpha_5,\delta_1,..,\delta_{g-2}\right\rangle = \langle\alpha_2,\alpha_5\rangle$, $\left\langle\alpha_3,\alpha_5,\delta_1,..,\delta_{g-2}\right\rangle = \langle\alpha_3,\alpha_5\rangle$, $\left\langle\alpha_1,\alpha_3,\delta_1,..,\delta_{g-2}\right\rangle = \langle\alpha_1,\alpha_3\rangle$ as in Figure 4.8, then the corresponding pentagon is denoted by $\langle\alpha_1,\alpha_4\rangle \leftrightarrow \langle\alpha_2,\alpha_4\rangle \leftrightarrow \langle\alpha_2,\alpha_5\rangle \leftrightarrow \langle\alpha_3,\alpha_5\rangle \leftrightarrow \langle\alpha_1,\alpha_3\rangle \leftrightarrow \langle\alpha_1,\alpha_4\rangle$.

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