ON THE SOLVABILITY OF A GROUP WITH PERMUTABLE SUBGROUPS

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# ABSTRACT


## ON THE SOLVABILITY OF A GROUP WITH PERMUTABLE SUBGROUPS

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It is well-known that a product of two solvable groups need not to be solvable. In this thesis, depending on an article of V. S. Monakhov [10], the solvability of a finite group $G = AB$ is studied. A subgroup $K$ of a group $G$ is called a Carter subgroup if $K$ is nilpotent and self-normalizing. A supersolvable subgroup $H$ of a group $G$ is called a Gaschütz subgroup if the condition $H \leq H_1 < T \leq G$ implies that $|T : H_1|$ is not prime. Using the Kegel-Wielandt and Kazarin results, Monakhov showed that if every Carter subgroup of $A$ commutes with every Carter subgroup of $B$, then $G = AB$ is solvable. Moreover, he gives that $G = AB$ is solvable when every Carter subgroup of $A$ is of odd order and commutes with every Gaschütz subgroup of $B$.

In addition, for convenience of the reader, the proofs of the properties of Carter subgroups given in the article "On nilpotent self-normalizing subgroups of solvable groups" of Roger. W. Carter are clarified.



Keywords: Finite soluble group, Sylow subgroup, Hall subgroup, Carter subgroup, Gaschütz subgroup.

# ÖZ

## Permutable ALTGRUPLU BİR GRUBUN ÇÖZÜLEBİLİRLİĞİ ÜZERİNE

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İki çözülebilir grubun çarpımının çözülebililir olmayabileceği bilinmektedir. Bu tezde, V. S. Monakhov'un makalesine dayanarak $G = AB$ tipindeki sonlu grubun çözülebilirliği çalışılmıştır. Bir $G$ grubunun nilpotent ve öz-normalleyen bir altgrubu var ise, bu altgruba $G$'nin Carter altgrubu denir. $G$ grubunun süperçözülebilir bir $H$ altgrubuna $H \leq H_1 < T \leq G$ iken $|T : H_1|$ asal değildir koşulunu sağlıyor ise $G$'nin Gashutz altgrubu denir. Monakhov, Kegel-Weiland ve Kazarin'nin sonuçlarını kullanarak gösteriyor ki eğer $A$'nın her Carter altgrubu, $B$'nin her Carter altgrubu ile degişmeli ise $G = AB$ çözülebilirdir. Ayrica $G = AB$'nin çözülebilirliğini $A$'nın her Carter altgrubunun tekil mertebeli ve $B$'nin her Gashutz altgrubu ile değişmeli olmasi koşulu altında da vermektedir.

Bunun yanı sıra, okuyucuya kolaylık sağlaması için tezde kullanılan Carter altgruplarının özellikleri Roger W. Carter'ın "On nilpotent self-normalizing subgroups of soluble groups" adlı makalesinden ispatları açıklanarak verilmiştir.

Anahtar Kelimeler: Sonlu çözülebilir grup, Sylow altgrup, Hall altgrup, Carter altgrup, Gaschütz altgrup

*To my family*

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# LIST OF SYMBOLS

| | | |
|---|---|---|
| $x^y$ | : | $y^{-1}xy$ where $x, y$ elements of a group |
| $[x, y]$ | : | $x^{-1}y^{-1}xy$ |
| $H \cong G$ | : | $H$ is isomorphic with $G$ |
| $H \leq G, H < G$ | : | $H$ is a subgroup, a proper subgroup of the group $G$. |
| $H \trianglelefteq G$ | : | $H$ is a normal subgroup of $G$ |
| $H \ sn \ G$ | : | $H$ is a subnormal subgroup of $G$ |
| $G^m, mG$ | : | Subgroup generated by all $g^m$ or $mg$ where $g \in G$ |
| $\|G\|$ | : | Cardinality of the group $G$ |
| $\|G : H\|$ | : | Index of the subgroup $H$ in the group $G$ |
| $C_G(H), N_G(H)$ | : | Centralizer, normalizer of $H$ in $G$ |
| $H^G$ | : | Normal closure of $H$ in $G$ |
| $H \times G$ | : | Direct product of $H$ and $G$ |
| $H \rtimes G$ | : | Semidirect product of $H$ with $G$ |
| $G' = [G, G]$ | : | Derived subgroup of a group $G$ |
| $V$ | : | Klein-four group |
| $D_{2n}$ | : | Dihedral group of order $2n$ |
| $\mathbb{Z}$ | : | Set of integers |
| $\mathbb{Z}_m$ | : | $\mathbb{Z}/m\mathbb{Z}$ |
| $\pi(G)$ | : | Set of prime divisors of the order of $G$ |
| $R(G)$ | : | Largest normal soluble subgroup of $G$ |
| $< H_1, H_2 >$ | : | Join of subgroups $H_1, H_2$ of a group $G$ |

# CHAPTER 1

## Introduction

In 1904, W. Burnside[8] showed that if $G$ is a finite group such that $|G|$ is divisible by at most two distinct primes, then $G$ is soluble. In the decade, 1928-1937, P. Hall[8] characterizes the soluble groups by means of the existence of Sylow complements. In particular, a finite group $G$ is soluble if and only if it is the product of pairwise permutable sylow subgroups. After, in 1958, Kegel-Wielandt[8] state that if the group $G = AB$ is the product of two nilpotent subgroups $A$ and $B$, then $G$ is soluble. Whereas, the group $G$ is nonsoluble if one of the factors is supersoluble. For example, the simple group of order 168 is the product of a supersoluble subgroup of order 21 and a nilpotent subgroup of order 8. In 1978, V. S. Monakhov[10] established that if the group $G = AB$ is the product of a supersoluble group $A$ with a group of odd order $B$ which is either primary or cyclic, then $G$ is soluble. More general results obtained by L. S. Kazarin[9] in 1980, says that if $A$ and $B$ are subgroups of the group $G$ such that $G = AB$, where $A$ is nilpotent of odd order, and $B$ is supersoluble, then $G$ is soluble.

The objective of this thesis is to give a detailed and clarified proof of the results of V. S. Monakhov[10] on the solubility of a group with permutable subgroups.

The following is a brief description of the thesis.

We consider finite groups only. Chapter 1 is the introduction of the thesis. Chapter 2 contains basic knowledge about several subjects in group theory that will be necessary in the subsequent chapter. Chapter 3 consists of three sections. In section 3.1 we study Carter subgroups. A subgroup $K$ of a group $G$ is called a Carter subgroup if $K$ is nilpotent and self-normalizing. In order to understand Carter subgroups thoroughly, we study the article of Roger. W. Carter[7] on nilpotent self-normalizing subgroups of soluble groups and clarified most of the results which were necessary for the main theorems. Also the results of V. S. Monakhov about Carter subgroups clarified. In particular, the following theorems were studied.

**Theorem 1.0.1** *[7]. Every finite soluble group $G$ possesses a Carter subgroup.*

**Lemma 1.0.2** *[7, 10]. Let $K$ be a Carter subgroup of a soluble group $G$. Then*

*a)* $K^x$ *is a Carter subgroup of $G$ for any* $x \in G$;

*b) if* $K \subseteq H \leq G$, *then $K$ is a Carter subgroup of* $H$;

*c) if* $K_1$ *is a Carter subgroup of $G$, then there is an element* $g \in G$ *such that* $K_1^g = K$;

*d) if $N$ is a normal subgroup of $G$, then $KN/N$ is a Carter subgroup of* $G/N$;

*e)* $K^G = G$.

**Lemma 1.0.3** *[10]. Let $G$ be a group, let $A$ be a soluble subgroup, let $N$ be a soluble normal subgroup of $G$. If $K$ is a Carter subgroup of $A$, then $KN/N$ is a Carter subgroup of $AN/N$.*

In section 3.2 we give another class of subgroups which is Gaschütz subgroups.

**Definition 1.0.4** *[3]. A subgroup $H$ of a group $G$ is called a Gaschütz subgroup if $H$ satisfies the following two conditions:*

*1) $H$ is supersoluble.*

*2) if* $H \leq H_1 < T \leq G$, *then* $|T : H_1|$ *is not prime.*

In section 3.3 the main results are given. In particular, in these sections the proofs of the following theorems are clarified.

**Theorem 1.0.5** *Let $A$ and $B$ be soluble subgroups of a group $G$ and* $G = AB$.

1) *If every Carter subgroup of A commutes with every Carter subgroup of B, then the group G is soluble.*

2) *If the Carter subgroups of A are of odd order and every Carter subgroup of A commutes with every Gaschütz subgroup of B, then the group G is soluble.*

**Theorem 1.0.6** *Let A and B be soluble subgroups of a group G and G = AB.*

1) *If every Sylow subgroup of A commutes with every Sylow subgroup of B, then the group G is soluble.*

2) *If every Sylow subgroup of A commutes with every Carter subgroup of B, then the group G is soluble.*

3) *If the subgroup A is of odd order and every Sylow subgroup of A commutes with every Gaschütz subgroup of B, then the group G is soluble.*

# CHAPTER 2

# Preliminaries

In this chapter we give the basic definitions and primary results that play an important role through other chapters.

## 2.1 Solvable Groups

In this section, the definition of the soluble group and its properties are given. Also it is given that the symmetric group $S_n$ on $n$ symbols is not soluble for $n \geq 5$.

**Definition 2.1.1** *[2]. Let $G$ be a group and*

$$\{e\} = H_0 \leq H_1 \leq H_2 \leq \cdots \leq H_n = G$$

*be a chain of subgroups of $G$. The chain is called a* **subnormal series** *if each $H_i$ is normal in $H_{i+1}$. The chain is called a* **normal series** *if each $H_i$ is normal in $G$. The number of proper inclusions $<$ in the chain is called the length of the chain.*

**Example 2.1.2** *[2].*

*i) The series*

$$\{(1)\} \leq \{(1), (12)(34)\} \leq V \leq A_4 \leq S_4$$

*is a subnormal series of the symmetric gruop $S_4$, where $V$ is the* **Klein-four group***. But it is not a normal series of $S_4$ since the subgroup $\{(1), (12)(34)\}$ is not normal in $S_4$.*

*ii) The series*

$$< [0] > \leq < [6] > \leq < [3] > \leq \ Z_{12}$$

*is a normal series of the abelian group $Z_{12}$.*

**Definition 2.1.3** *[6]. Let $H \leq G$. If there exists a finite chain of subgroups $H = H_0 \leq H_1 \leq H_2 \leq \cdots H_n = G$ where each $H_i$ is normal in $H_{i+1}$, then $H$ is called a* **subnormal subgroup** *of $G$ and it is denoted by $H$ sn $G$.*

**Definition 2.1.4** *[1]. A group $G$ is said to be* **soluble** *if it has an abelian series, by which we mean a series*

$$\{e\} = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_n = G$$

*in which each factor $G_{i+1}/G_i$ is abelian.*

**Definition 2.1.5** *[1]. If $G$ is a soluble group, the length of a shortest abelian series in $G$ is called the derived length of $G$.*

**Example 2.1.6** *Let us give some examples.*

*i) Every abelian group is soluble of derived length 1.*

*ii) The non-abelian symmetric group $S_3$ is soluble of derived length 2. Note that*

$$\{(1)\} \triangleleft A_3 \triangleleft S_3$$

*is a series of $S_3$ such that $S_3/A_3 \cong Z_2$ and $A_3/\{(1)\}$ are abelian.*

*iii) The symmetric group $S_4$ is soluble as*

$$\{(1)\} \triangleleft V \triangleleft A_4 \triangleleft S_4$$

*is an abelian series, where $V$ is the* **Klein-four group**.

*iv) For any $n \geq 2$, the Dihedral group $D_{2n} = C_n \rtimes C_2$ having abelian series*

$$\{1\} \triangleleft C_n \triangleleft D_{2n}$$

*is soluble.*

**Definition 2.1.7** *[1].*

*i) The group $G^{'} = [G, G] = \langle [a, b] : a, b \in G \rangle$ where $[a, b] = a^{-1}b^{-1}ab$ is called the* **derived subgroup** *of G or (the commutator subgroup of G).*

*ii) The descending series*

$$G = G^{(0)} \geq G^{'} \geq G^{(2)} \geq G^{(3)} \geq \cdots \geq G^{(n)} \geq \cdots$$

*is called the* **derived series** *of G where $G^{(n+1)} = (G^{(n)})^{'}$.*

**Remark 2.1.8** *The derived series need not to terminate after finitely many steps or need not to reach $\{e\}$.*

**Theorem 2.1.9** *[2]. The derived subgroup $G^{'}$ of a group G is a normal subgroup of G and $G/G^{'}$ is commutative.*

**Theorem 2.1.10** *[2]. Let $G^{'}$ be the derived subgroup of a group G and H be a subgroup of G. Then $G^{'} \leq H$ if and only if H is a normal subgroup of G and $G/H$ is commutative.*

**Proof.** Assume that $G^{'} \leq H$. Take any element $h \in H$ and $g \in G$. Then

$$h^{g} = g^{-1}hg = g^{-1}hgh^{-1}h = [g, h^{-1}]h$$

where $[g, h^{-1}] \in G^{'} \leq H$. So, $h^{g} \in H$ and $H$ is a normal subgroup of $G$. Now we want to show that $G/H$ is commutative. Let us take any two arbitrary elements $aH, bH \in G/H$. Since $[a, b] \in G^{'} \leq H$, it follows that $[a, b]H = H$. That is $a^{-1}b^{-1}abH = H$. Then, $aHbH = bHaH$. Hence, $G/H$ is commutative. Conversely, assume that $H$ is normal in $G$ and $G/H$ is commutative. Take any $a, b \in G$. Since $G/H$ is commutative we have $aHbH = bHaH$. Then $a^{-1}b^{-1}abH = H$ which implies that $a^{-1}b^{-1}ab \in H$. Hence, $G^{'} \leq H$. □

**Theorem 2.1.11** *[2]. Let G be a group. Then G is soluble if and only if there is a positive integer m such that $G^{(m)} = \{e\}$.*

**Proof.** Assume that $G^{(m)} = \{e\}$ for some positive integer $m$. Then by Theorem 2.1.9, the series

$$G = G^0 \geq G^{'} \geq G^{(2)} \geq \cdots \geq G^{(m)} = \{e\}$$

is a subnormal abelian series of $G$. So, $G$ is soluble. Conversely, assume that $G$ is soluble. Then $G$ has a subnormal abelian series, say,

$$\{e\} = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_m = G.$$

Since $G_i$ is normal in $G_{i+1}$ and $G_{i+1}/G_i$ is commutative, by Theorem 2.1.10, the commutator subgroup $G^{'}_{i+1}$ is contained in $G_i$ for all $i$. Thus,

$$G^{(m)} = G^{(m)}_m \leq G^{(m-1)}_{m-1} \leq G^{(m-2)}_{m-2} \leq \cdots \leq G^{'}_1 \leq G_0 = \{e\}.$$

Hence, $G^{(m)} = \{e\}$. □

**Lemma 2.1.12** *[1]. $A_n$ is generated by 3-cycles if $n \geq 3$.*

**Lemma 2.1.13** *[2]. Let $S_n$ be the symmetric group on n symbols. If $n \geq 5$, then $S^{(k)}_n$ contains every 3-cycle of $S_n$ for $k = 1, 2, \ldots$.*

**Theorem 2.1.14** *[2]. The symmetric group $S_n$ on n symbols is not soluble for $n \geq 5$.*

**Proof.** Let $S_n$ be the symmetric group on $n$ symbols with $n \geq 5$. Then by Lemma 2.1.13, $S^{(k)}_n$ contains every 3-cycle of $S_n$ for, $k = 1, 2, \ldots$. By Lemma 2.1.12, $A_n \leq S^{(k)}_n$ for $k \geq 1$. And so there does not exist a positive integer $m$ such that $S^{(m)}_n = \{e\}$. Thus, by Theorem 2.1.11, $S_n$ is not soluble. □

**Theorem 2.1.15** *[1]. The class of soluble groups is closed with respect to the formation of subgroups, images, and extensions of its members.*

**Proof.** Let $G$ be a soluble group and $H \leq G$. Let

$$\{e\} = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_n = G.$$

be an abelian series of $G$. Set $H_i = H \cap G_i$. Then

$$\{e\} = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_n = H.$$

Since $G_i \lhd G_{i+1}$, it follows that $H \cap G_i \lhd H \cap G_{i+1}$ and so $H_i \lhd H_{i+1}$. Now,

$$H_{i+1}/H_i = H \cap G_{i+1}/H \cap G_i = H \cap G_{i+1}/(H \cap G_{i+1}) \cap G_i$$

and by the Second Isomorphism Theorem , we have

$$H \cap G_{i+1}/(H \cap G_{i+1}) \cap G_i \cong (H \cap G_{i+1})G_i/G_i.$$

The group $(H \cap G_{i+1})G_i/G_i$ is abelian since it is a subgroup of the abelian group $G_{i+1}/G_i$. Hence, $H_{i+1}/H_i$ is abelian and then $H$ is soluble.
Let $N$ be a normal subgroup of $G$. Consider the series

$$N/N = G_0N/N \lhd G_1N/N \lhd G_2N/N \lhd \cdots \lhd G_nN/N = G/N.$$

Since $G_i \lhd G_{i+1}$, we get $G_iN/N \lhd G_{i+1}N/N$. Now, by the Second and Third Isomorphism Theorems,

$$(G_{i+1}N/N)/(G_iN/N) \cong G_{i+1}N/G_iN = G_{i+1}(G_iN)/G_iN$$

which is isomorphic to

$$G_{i+1}/G_{i+1} \cap (G_iN) \cong (G_{i+1}/G_i)/(G_{i+1} \cap (G_iN)/G_i).$$

Since $G_{i+1}/G_i$ is abelian, the homomorphic image $(G_{i+1}/G_i)/(G_{i+1} \cap (G_iN)/G_i)$ is abelian and so

$$(G_{i+1}N/N)/(G_iN/N)$$

is abelian. Hence, $G/N$ is soluble.
Assume that $N$ and $G/N$ are soluble groups. Consider the abelian serieses

$$\{e\} = N_0 \lhd N_1 \lhd N_2 \lhd \cdots \lhd N_m = N$$

and

$$N/N = G_0/N \lhd G_1/N \lhd G_2/N \lhd \cdots \lhd G_n/N = G/N.$$

of $N$ and $G/N$, respectively. We must show

$$\{e\} = N_0 \leq N_1 \leq N_2 \leq \cdots \leq N_m = N \leq G_1 \leq G_2 \leq \cdots \leq G_n = G$$

is an abelian series of $G$. Now $G_i/N \lhd G_{i+1}/N$ implies that $G_i \lhd G_{i+1}$. By the Third Isomorphism Theorem

$$G_{i+1}/G_i \cong (G_{i+1}/N)/(G_i/N)$$

and so $G_{i+1}/G_i$ is abelian. Also since $N$ is soluble $N_{i+1}/N_i$ is abelian. So the above series is an abelian series of $G$. Hence $G$ is soluble. □

**Theorem 2.1.16** *[1]. The product of two normal soluble subgroups of a group is soluble.*

**Proof.** Let $M$ and $N$ be two normal soluble subgroups of a group $G$. By the Second Isomorphism Theorem, we have

$$MN/N \cong M/M \cap N.$$

Since $M$ is soluble, by Theorem 2.1.15 the homomorphic image $M/M \cap N$ is soluble. So, $MN/N \cong M/M \cap N$ is soluble. Then by Theorem 2.1.15, $MN$ is soluble as, $MN/N$ and $N$ are soluble. □

**Definition 2.1.17** *[5]. A subgroup $N$ of a group $G$ is called* **minimal normal subgroup** *of $G$ if it is a nontrivial normal subgroup of $G$, and if $M \triangleleft G$ and $\{e\} \leq M \leq N$, then either $M = \{e\}$ or $M = N$. If the group G is simple, then the only minimal normal subgroup of G is G itself.*

**Example 2.1.18** *The alternating group $A_3$ is the minimal normal subgroup of the symmetric group $S_3$.*

**Definition 2.1.19** *[5]. An abelian group G in which every nontrivial element has order p, for some fixed prime $p$, is called an* **elementary abelian p-group**.

**Theorem 2.1.20** *[5]. If G is a finite soluble group and N is a minimal normal subgroup of $G$, then N is an elementary abelian p-group for some prime $p$.*

**Example 2.1.21** *[5]. The Klein-four group V is a minimal normal subgroup of the soluble group $S_4$, and $V \cong Z_2 \times Z_2$ is elementary abelian. Note that Theorem 2.1.20 is true for soluble groups. But in general for non-soluble groups it is not true , for example, $A_5$ is the only minimal normal subgroup of the symmetric group $S_5$ that it is not elementary abelian.*

## 2.2 Supersoluble group

**Definition 2.2.1** *[6]. A group G is called supersoluble if there exists a series*

$$\{e\} = G_0 \leq G_1 \leq \cdots \leq G_{n-1} \leq G_n = G$$

*such that* $G_i \triangleleft G$ *and* $G_{i+1}/G_i$ *is cyclic for each i,* $0 \leq i \leq n-1$.

**Example 2.2.2** *The non-abelian symmetric gruop* $S_3$ *is supersoluble.*

**Example 2.2.3** *The symmetric group* $S_4$ *is a soluble group that is not supersoluble. Note that the series*

$$\{(1)\} \leq \{(1), (12)(34)\} \leq V \leq A_4 \leq S_4$$

*is not a normal series of* $S_4$ *since the subgroup* $\{(1), (12)(34)\}$ *is not normal in* $S_4$.

**Theorem 2.2.4** *[6]. Let G be a supersoluble group. Then the following hold.*

*i) Any subgroup of G is supersoluble.*

*ii) Every homomorphic image of G is supersoluble.*

### 2.3 Sylow Theorems

**Definition 2.3.1** *[2]. Let p be a prime. A group G is said to be a* **p-group** *if the order of each element of G is a power of p. A subgroup H of a group G is called a* **p-subgroup** *if H is a p-group.*

**Example 2.3.2** *[2]. The Klein-four group, is a 2-group.*

**Theorem 2.3.3** *[2]. If G is a finite p-group with* $|G| > 1$, *then* $Z(G)$, *the center of* $G$, *has more than one element, i.e., if* $|G| = p^k$ *with* $k \geq 1$, *then* $|Z(G)| > 1$.

**Definition 2.3.4** *[2]. Let G be a finite group and p a prime. A subgroup P of G is called a* **Sylow p-subgroup** *of* $G$, *if P is a p-subgroup and is not properly contained in any other p-subgroup of* $G$, *i.e., P is a maximal p-subgroup of* $G$.

**Example 2.3.5** *[2]. The symmetric group* $S_3$ *has the following Sylow 2-subgroups,*

$$K_1 = \{(1), (12)\},$$

$$K_2 = \{(1), (13)\},$$

$$K_3 = \{(1), (23)\}.$$

**Theorem 2.3.6** *(Sylow's First Theorem)[2]. Let G be a finite group of order* $p^r m$, *where p is a prime, r and m are positive integers, p and m are relatively prime. Then G has a subgroup of order* $p^k$ *for all* $k$, $0 \leq k \leq r$.

**Theorem 2.3.7** *(Sylow's Second Theorem)[2]. Let G be a finite group of order* $p^r m$, *where p is a prime, r and m are positive integers, p and m are relatively prime. Then any two Sylow p-subgroups of G are conjugate, and therefore isomorphic.*

### 2.4 Hall $\pi$-subgroup

**Definition 2.4.1** *[1].*

i) *A* **Hall subgroup** *of a finite group G is subgroup H such that* $|H|$ *and* $|G : H|$ *are relatively prime.*

ii) *A subgroup H of a finite group G is called a* **Hall $\pi$-subgroup** *(where $\pi$ is a set of primes) if* $|H|$ *is a product of primes in $\pi$ and* $|G : H|$ *is divisible by no prime in $\pi$.*

**Example 2.4.2** $S_4$ *is a Hall $\pi$-subgroup of* $S_5$*, where* $\pi = \{2, 3\}$ .

**Theorem 2.4.3** *(P. Hall)[1]. If G is a finite soluble group, then every $\pi$-subgroup is contained in a Hall $\pi$-subgroup of G. Moreover all Hall $\pi$-subgroups are conjugate in G.*

**Definition 2.4.4** *[6]. Let* $K \trianglelefteq G$*. We say that G splits over K if there is a subgroup H of G such that* $G = HK$ *and* $H \cap K = 1$*. Any such subgroup H is said to be a* **complement to** *K in G.*

**Example 2.4.5** *The symmetric group* $S_3$ *splits over* $A_3$ *since* $A_3 \cap < (13) >= 1$ *and* $S_3 = A_3 < (13) >$ .

**Definition 2.4.6** *[6]. Let G be a finite group. A complement to a Sylow p-subgroup of G is called a* **p-complement of** *G. Note that a subgroup H of G is a p-complement of G if and only if* $|G : H|$ *is a power of p and p does not divide* $|H|$ *. In particular, a p-complement of G is a* **Hall $p'$-subgroup** *of G (where $p'$ denotes the set of all primes distinct from p).*

## 2.5 Nilpotent Groups

This section gives elementary properties of nilpotent groups.

**Definition 2.5.1** *[1]. A group $G$ is called* **nilpotent** *if it has a* **central series***, that is, a normal series*

$$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G,$$

*such that $G_{i+1}/G_i$ is contained in the center of $G/G_i$ for all $i$. The length of a shortest central series of $G$ is called the* **nilpotent class** *of $G$.*

**Example 2.5.2** *[2]. The symmetric group $S_3$ has only two normal series, $\{e\} \leq S_3$ and $\{e\} \leq A_3 \leq S_3$. For the first series, $S_3/\{e\} \nsubseteq Z(S_3/\{e\})$. For the second series, $A_3/\{e\} \nsubseteq Z(S_3/\{e\})$. Hence, $S_3$ is not a nilpotent group. However, $S_3$ is soluble.*

**Theorem 2.5.3** *(Kegel and Wielandt)[8]. If the group $G = HK$ is the product of two nilpotent subgroups $H$ and $K$, then $G$ is soluble.*

**Definition 2.5.4** *[2]. For any group $G$, define $Z_0(G) = \{e\}$, $Z_1(G) = Z(G)$ and $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$, for $i \geq 1$. Then the series*

$$\{e\} \leq Z(G) \leq Z_2(G) \leq \cdots \leq Z_i(G) \leq \cdots$$

*is called the* **upper central series** *of $G$.*
*Now, let $L_1(G) = G$, $L_2(G) = [G, G] = G'$, $\ldots, L_{k+1}(G) = [L_k(G), G]$. Then the series*

$$G = L_1 \geq L_2 \geq \cdots \geq L_k \geq L_{k+1} \geq \cdots$$

*is called the* **lower central series** *of $G$.*

**Theorem 2.5.5** *[1]. A group is nilpotent if and only if the lower central series reaches the identity subgroup after a finite number of steps or, equivalently, the upper central series reaches the group itself after a finite number of steps.*

**Theorem 2.5.6** *[2]. Every finite $p$-group is nilpotent.*

**Proof.** Let $G$ be a finite p-group. If $G = \{e\}$, then nothing to prove. Assume that $G \neq \{e\}$. Then by Theorem 2.3.3, we have $Z_1 = Z(G) \neq \{e\}$. If $Z_1 \neq G$, then $G/Z_1$ is a nontrivial finite p-group. Thus by Theorem 2.3.3, we have $|Z(G/Z_1)| > 1$. Set $Z_2/Z_1 = Z(G/Z_1)$. Then $Z_2/Z_1$ is a nontrivial group, where $Z_2$ is a normal subgroup of $G$. Therefore, we obtain the following chain

$$\{e\} < Z_1 < Z_2.$$

If $Z_2 \neq G$, we continue the above process and we get the normal series

$$\{e\} < Z_1 < Z_2 < Z_3$$

where $Z_3/Z_2 = Z(G/Z_2)$. Since $G$ is finite, this process must stop after finitely many steps. That is there exists a positive integer $n$ such that $Z_n(G) = G$ and we get the upper central series

$$\{e\} < Z_1 < Z_2 < Z_2 \cdots Z_n(G) = G$$

where $Z_{i+1}/Z_i = Z(G/Z_i)$. Therefore by Theorem 2.5.5, $G$ is nilpotent. □

**Lemma 2.5.7** *[6]. A series of $G$, say*

$$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G,$$

*is a central series if and only if, $[G_i, G] \leq G_{i-1}$ for each $i = 1, \ldots, n$.*

**Proof.** Suppose that

$$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G\ ;$$

is a central series then, $G_{i-1} \trianglelefteq G$ and $G_i/G_{i-1} \leq Z(G/G_{i-1})$, for each $i = 1, \ldots, n$. Take $y \in G$ and $x \in G_i$. Then, $y\, G_{i-1} \in G/G_{i-1}$ and $x\, G_{i-1} \in G_i/G_{i-1} \leq Z(G/G_{i-1})$. So, $(x\, G_{i-1})(y\, G_{i-1}) = (y\, G_{i-1})(x\, G_{i-1})$. Then $x^{-1}y^{-1}x\, y\, G_{i-1} = G_{i-1}$. Therefore,

$$[x, y] = x^{-1}y^{-1}x\, y \in G_{i-1}.$$

That is

$$[G_i, G] \leq G_{i-1}.$$

Conversely, suppose that $[G_i, G] \leq G_{i-1}$ for each $i = 1, \ldots, n$. Let $y \in G$ and $x \in G_{i-1}$. Since $G_{i-1} \leq G_i$ we get $x \in G_i$. Then,

$$x^{-1}x^{y} = x^{-1}y^{-1}x\, y \in [G_i, G] \leq G_{i-1}.$$

Thus $x^y = xx^{-1}x^y \in G_{i-1}$. That is $G_{i-1} \trianglelefteq G$. Also, $x^{-1}y^{-1}x\,y\,G_{i-1} = G_{i-1}$ as $x^{-1}y^{-1}x\,y \in G_{i-1}$. Thus,

$$(x\,G_{i-1})(y\,G_{i-1}) = (y\,G_{i-1})(x\,G_{i-1}),$$

and then

$$G_i/G_{i-1} \leq Z(G/G_{i-1}).$$

Hence the series is central. □

**Theorem 2.5.8** *[6]. If G is nilpotent then all subgroups and all quotient groups of G are nilpotent.*

**Proof.** Let $G$ be nilpotent group with a central series

$$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G. \tag{2.1}$$

Let $H \leq G$ and $K \trianglelefteq G$. Then,

$$\{e\} = (G_0 \cap H) \trianglelefteq (G_1 \cap H) \trianglelefteq \cdots \trianglelefteq (G_n \cap H) = H, \tag{2.2}$$

and

$$K/K = G_0K/K \trianglelefteq G_1K/K \trianglelefteq \cdots \trianglelefteq G_nK/K = G/K. \tag{2.3}$$

Since (2.1) is a central series of $G$, by Lemma 2.5.7, we obtain $[G_i, G] \leq G_{i-1}$, for each $i = 1, \ldots, n$. Hence

$$[G_i \cap H, H] \leq H \cap [G_i, G] \leq H \cap G_{i-1},$$

and

$$[G_iK/K, G/K] = [G_i, G]K/K \leq G_{i-1}K/K.$$

Thus, by Lemma 2.5.7, (2.2) and (2.3) are central series, and then $H$ and $G/K$ are nilpotent. □

**Theorem 2.5.9** *[2]. Let G be a finite group. Then the following conditions are equivalent.*

*i) G is nilpotent.*

*ii) If H is a proper subgroup of G, then* $H \subset N_G(H)$. *(A group with this property is said to satisfy the normalizer condition.)*

*iii) Every maximal subgroup of G is a normal subgroup of G.*

*iv) Every Sylow subgroup of G is a normal subgroup of G.*

*v) G is isomorphic to a direct product of its Sylow subgroups.*

**Theorem 2.5.10** *[2]. Let* $G_i$, $i = 1, 2, \ldots, n$, *be a nilpotent group. Then*

$$G_1 \times G_2 \times \cdots \times G_n$$

*is nilpotent.*

**Corollary 2.5.11** *Let G be a finite nilpotent group. Then for any prime* $p$. *G has a unique Sylow* $p$*-subgroup.*

# CHAPTER 3

# On the Solvability of a Group with Permutable Subgroups

## 3.1 Carter Subgroups

**Definition 3.1.1** *[3]. A subgroup $K$ of a group $G$ is called a* **Carter subgroup** *if $K$ is nilpotent and self-normalizing.*

**Definition 3.1.2** *[7]. $H$ is called an* **abnormal subgroup** *of $G$ if $g \in < H, g^{-1}Hg >$ for every element $g \in G$.*

**Lemma 3.1.3** *[7]. $H$ is abnormal in $G$ if and only if it satisfies the following two conditions:*

*i) Every subgroup of $G$ containing $H$ is its own normalizer.*

*ii) $H$ is not contained in two different conjugate subgroups of $G$.*

**Proof.** Suppose $H$ is abnormal in $G$.

i) Let $H \leq L$ and let $g \in N_G(L)$. Then $L = g^{-1}Lg$ and so,

$$g \in < H, g^{-1}Hg > \leq < L, g^{-1}Lg >= L.$$

Hence, $L$ is its own normalizer.

ii) Let $H \leq L \cap gLg^{-1}$ for some $g \in G$. So, $g^{-1}Hg \leq g^{-1}Lg \cap L \leq L$ and we get $g \in < H, g^{-1}Hg > \leq L$. Therefore $gLg^{-1} = L$ and $H$ is not contained in two different conjugate subgroups of $G$.

Conversely, suppose that $H$ satisfies the above two conditions. Take any $g \in G$ and set $L =< H, g^{-1}Hg >$ . Since $g^{-1}Hg \leq L$, we obtain $H \leq gLg^{-1}$. Then, $H \leq L \cap gLg^{-1}$. By condition (ii), $L = gLg^{-1}$. Hence $g$ is in the normalizer of $L$ which is $L$ itself by condition (i). Thus, $g \in< H, g^{-1}Hg >$ for all $g \in G$. Hence, $H$ is an abnormal subgroup of $G$. □

**Lemma 3.1.4** *[7]. Let G be soluble group and H be the normalizer of a Hall subgroup S of G. Then H is abnormal in G.*

**Proof.** Take any $g \in G$. Since $S$ and $g^{-1}Sg$ have the same order, $g^{-1}Sg$ is also a Hall subgroup of $G$. Set $L =< H, g^{-1}Hg >$ . Then $S$ and $g^{-1}Sg$ are also Hall subgroups of $L$. By Theorem 2.4.3, there exists $l \in L$ such that $g^{-1}Sg = l^{-1}Sl$. Thus $gl^{-1} \in N_G(S) = H \leq L$ and hence $g \in L$. Therefore $H$ is abnormal in $G$. □

**Theorem 3.1.5** *[7]. Every finite soluble group G possesses a Carter subgroup.*

**Proof.** We will prove by induction on the order of $G$. Let $N$ be a minimal normal subgroup of the soluble group $G$. Then by Theorem 2.1.20, $N$ is an elementary abelian p-group for some prime $p$. Since $|G/N| < |G|$, by induction assumption $G/N$ has a Carter subgroup say $K/N$. As $K/N$ is nilpotent, by Corollary 2.5.11, it has a unique Sylow p-subgroup say $P/N$. Since $N$ is a p-group, so is $P$. Take any Sylow p-subgroup $Q$ of $K$. Then $QN/N$ is a p-group of $K/N$. Thus by Definition 2.3.4, $QN/N \leq P/N$. That is $Q \leq P$ and so $Q = P$. Hence, $K$ has a unique Sylow p-subgroup $P$. Let $S$ be a Sylow p-complement (equivalently, a Hall $p'$-subgroup) of $P$ in $K$. That is $K = PS$ and $P \cap S = \{e\}$. Set $H = N_K(S)$. We claim that $H$ is a Carter subgroup of $G$. Put $P_1 = H \cap P$. Then $P_1S = (H \cap P)S$ and by Dedekind identity,

$$(H \cap P)S = H \cap PS = H \cap K = H.$$

Since $P$ is normal in $K$, so $P_1$ is normal in $H$. Moreover, $P_1$ is a p-group of $H$ and by the Second Isomorphism Theorem,

$$|H : P_1| = |H : H \cap P| = |HP : P| = |K : P|$$

is relatively prime to $p$ (note that the last equality holds as $K = SP \leq HP \leq K$ implies $K = HP$.) Thus $P_1$ is the unique Sylow p-subgroup of $H$. Also $S \trianglelefteq H$ by the definition

$H = N_K(S)$, and so $S$ is a Sylow p-complement of $H$. Hence, $H = P_1 \times S$. But,

$$K/P = SP/P \cong S/S \cap P \cong S.$$

By the Third Isomorphism Theorem, $K/P \cong (K/N)/(P/N)$. Since $K/P$ is the homomorphic image of the nilpotent group $K/N$, by Theorem 2.5.8, it is nilpotent. Thus, $S$ is nilpotent and so by Theorem 2.5.10, $H$ is nilpotent.
Since $H$ is the normalizer of the Sylow p-complement (Hall $p'$-subgroup) $S$ of $K$, by Lemma 3.1.4, $H$ is abnormal in $K$. So, by Lemma 3.1.3 part (i), $N_K(NH) = NH$. Thus $N_{K/N}(NH/N) = NH/N$. But the group $K/N$ is nilpotent. Thus, by Theorem 2.5.9 part (ii), it satisfies the normalizer condition, and we get $NH = K$. Take any $g \in N_G(H)$, then $H^g = H$. Now $N$ is normal in $G$, then $N^g = N$ for any $g \in G$. So,

$$K^g = (NH)^g = NH = K.$$

Hence, $g \in N_G(K)$. So, $(K/N)^{gN} = K^g/N = K/N$. But $N_{G/N}(K/N) = K/N$. Then $g \in K$. Therefore, $g \in N_K(H)$. By the abnormality of $H$ in $K$, see Lemma 3.1.3 part (i), we get $N_K(H) = H$. Thus, $g \in H$ and $H$ is self-normalizing subgroup of $G$. Hence, $H$ is a Carter subgroup of the soluble group $G$. □

**Example 3.1.6** *The soluble group $S_3$ has the Carter subgroups:*

$$K_1 = \{(1), (12)\},$$

$$K_2 = \{(1), (13)\},$$

$$K_3 = \{(1), (23)\}.$$

**Lemma 3.1.7** *Let K be a Carter subgroup of a soluble group G. Then*

*a) $K^x$ is a Carter subgroup of G for any $x \in G$;*

*b) if $K \subseteq H \leq G$, then K is a Carter subgroup of H;*

*c) if $K_1$ is a Carter subgroup of G, then there is an element $g \in G$ such that $K_1^g = K$;*

*d) if N is a normal subgroup of G, then $KN/N$ is a Carter subgroup of $G/N$;*

*e)* $K^G = G$.

**Proof.**

a) Take an arbitrary element $g \in N_G(K^x)$, then $(K^x)^g = K^x$ which gives $xg^{-1}x^{-1}Kxgx^{-1} = K$. So, $xgx^{-1} \in N_G(K) = K$. That is $xgx^{-1} = k$ for some $k \in K$. Hence,

$$g = x^{-1}kx = k^x \in K^x.$$

Thus, $N_G(K^x) = K^x$. Also, $K^x$ is nilpotent as $K$ is so. Therefore, $K^x$ is a Carter subgroup of $G$.

b) By definition of Carter subgroup, $K$ is nilpotent and $N_G(K) = K$. Then,

$$K \leq N_H(K) \leq N_G(K) = K.$$

Thus, we have $N_H(K) = K$. Therefore $K$ is a Carter subgroup of $H$.

c) See [7]. We will use induction on the order of $G$. Let us first show that every Carter subgroup of $G$ is abnormal. Let $H$ be a Carter subgroup of $G$ and $g \in G$. Set $L =< H, g^{-1}Hg >$ . We show that $g \in L$. If $L = G$, then trivially $g \in L$. If $L < G$, then $H$ and $g^{-1}Hg$ are Carter subgroups of $L$. Hence, by inductive hypothesis they are conjugate in $L$; i.e., $g^{-1}Hg = l^{-1}Hl$ for some $l \in L$. Therefore $gl^{-1} \in N_G(H) = H \leq L$ and so $g \in L =< H, g^{-1}Hg >$ . Hence, $H$ is abnormal in $G$.

Let $N$ be a normal subgroup of $G$. Then by Lemma 3.1.3 part (i), $NH$ is its own normalizer in $G$. So, $NH/N$ is its own normalizer in $G/N$. Also, $NH/N$ is nilpotent as by the Second Isomorphism Theorem it is isomorphic to the homomorphic image $H/N \cap H$ of the nilpotent group $H$. So, $NH/N$ is a Carter subgroup of $G/N$.

Let $K$ and $K_1$ be two Carter subgroups of $G$ and $N$ be a minimal normal subgroup of the soluble group $G$. Then by Theorem 2.1.20, $N$ is an elementary abelian p-group for some prime $p$. Also, $KN/N$ and $K_1N/N$ are Carter subgroups of $G/N$. Since $|G/N| < |G|$, by induction assumption there exists $g_1N \in G/N$ such that

$$KN/N = (K_1N/N)^{g_1N} = g_1^{-1}N(K_1N/N)g_1N = g_1^{-1}K_1g_1N/N = K_1^{g_1}N/N.$$

Then $KN = K_1^{g_1}N$. Assume that $KN < G$. Then $K_1^{g_1} \leq K_1^{g_1}N = KN < G$ and by part (a) and (b), we get $K_1^{g_1}$ is a Carter subgroup of $KN$. Similarly, $K$ is a Carter subgroup of $KN$. Therefore, by induction there exists $g_2 \in KN$ such that $(K_1^{g_1})^{g_2} = K$, that is $K_1^g = K$ for some $g = g_1g_2$. Now, assume that $KN = K_1^{g_1}N = G$. As $N \lhd G$ we get $N \cap K \lhd K$. Also, since $N$ is abelian, we have $N \cap K \lhd N$. Hence, $N \cap K \lhd KN = G$. Therefore, either $N \cap K = N$ or $N \cap K = 1$. Similarly, either $N \cap K_1^{g_1} = N$ or $N \cap K_1^{g_1} = 1$. If $N \cap K = N$, then $N \leq K$ and so $G = KN = K$ which is the trivial case. Also $N \cap K_1^{g_1} = N$, we get the trivial case $G = K_1^{g_1}$. Suppose that $N \cap K = 1$ and $N \cap K_1^{g_1} = 1$. Then $K$ and $K_1^{g_1}$ are complements of $N$ in $G$. Since $N$ is an elementary abelian p-group, $|N| = p^{\alpha}$ for some positive integer $\alpha$. Now $|G| = |KN| = |K|\,|N| \,/\, |K \cap N| = |K|\,|N|$ and also by Lagrange Theorem we have $|G| = |G : K|\,|K|$. So $|G : K| = |N| = p^{\alpha}$. Similarly, $\left|G : K_1^{g_1}\right| = |N| = p^{\alpha}$. Note that $K$ is a maximal subgroup of $G$. Suppose that there exists a subgroup $M$ such that $K < M < G$. Then, $KN \leq MN \leq G$ and as $KN = G$, we get $MN = G$. Then, as $N$ is abelian and normal in $G$ we get that $N \cap M \lhd M$, and $N \cap M \lhd N$. Hence, $N \cap M \lhd MN = G$. But $1 < N \cap M < N$ which is a contradiction, as $N$ is the minimal normal subgroup of $G$. Hence, $K$ is a maximal subgroup of $G$. Similarly, one may also see that $K_1^{g_1}$ is a maximal subgroup of $G$.

Let $S$ and $S_1$ be the Sylow p-complements (Hall $p'$-subgroups) of $K$ and $K_1^{g_1}$ respectively. Then $|G : S| = |G : K|\,|K : S|$ and $|G : S_1| = \left|G : K_1^{g_1}\right|\left|K_1^{g_1} : S_1\right|$ are powers of $p$ and so $S$ and $S_1$ are Sylow p-complements of $G$. By Hall's Theorem, see 2.4.3, they are conjugate in $G$. That is, there exists $g \in G$ such that $S = g^{-1}S_1g$. Assume that $K \neq g^{-1}(K_1^{g_1})g$. Now, $K = P \times S$ and $K_1^{g_1} = Q \times S_1$ where $P$, $Q$ are the Sylow p-subgroups of the nilpotent groups $K$ and $K_1^{g_1}$. Then, $g^{-1}K_1^{g_1}g = g^{-1}Qg \times g^{-1}S_1g = g^{-1}Qg \times S$ which means $S$ is normal in the groups $K$ and $g^{-1}(K_1^{g_1})g$. Therefore,

$$S \lhd \, < K, g^{-1}(K_1^{g_1})g >= G,$$

as $K$ is maximal subgroup of $G$. Now, $G/S$ is a finite p-group and by Theorem 2.5.6, it is nilpotent. Take any element $gS \in N_{G/S}(K/S)$. Then $K/S = (K/S)^{gS} = K^g/S$ and so $K^g = K$. It means that $g \in N_G(K) = K$. Thus $gS \in K/S$ and we obtain $K/S = N_{G/S}(K/S)$, which is a contradiction since $G/S$ satisfies the normalizer condition by theorem 2.5.9 part 2. Hence, $K = g^{-1}(K_1^{g_1})g$ and $K, K_1^{g_1}$ are conjugate in $G$.

d) By Second Isomorphism Theorem $KN/N \cong K/K \cap N$ and as the homomorphic image of $K$ is nilpotent by Theorem 2.5.8, the group $K/K \cap N$ is nilpotent. Then $KN/N$ is nilpotent. Now, to show that $N_{G/N}(KN/N) = KN/N$, take an arbitrary element $xN \in N_{G/N}(KN/N)$. Then

$$KN/N = (KN/N)^{xN} = K^x N/N$$

and so $KN = K^x N$. Then, we have $K \leq KN \leq G$ and $K^x \leq K^x N = KN \leq G$. Then, $K$ and $K^x$ are Carter subgroups of $KN$, by part (b). Also since $KN$ is a subgroup of the soluble group $G$, then $KN$ is soluble. So by part (c) they are conjugate in $KN$. Thus, there is an element $y \in KN$ such that $K = (K^x)^y$ and so $xy \in N_G(K) = K$. It follows that, $xy = k$ for some $k \in K$. Then, $x = ky^{-1} \in KN$, and, $xN \in KN/N$. Hence, $N_{G/N}(KN/N) = KN/N$.

e) Let $x$ be an element in $G$. Then, $K^x < K^G \leq G$. Then $K$ and $K^x$ are Carter subgroups of $K^G$ by part (b). Hence, by part (c), they are conjugate in $K^G$. That is there is an element $y \in K^G$ such that $K = (K^x)^y$. So $xy \in N_G(K) = K$ which implies that $x \in Ky^{-1} \subseteq K^G$. Thus, $G = K^G$.

□

**Lemma 3.1.8** *Let G be a group, let A be a soluble subgroup, let N be a soluble normal subgroup of G. If K is a Carter subgroup of A, then KN/N is a Carter subgroup of AN/N.*

**Proof.** By Second Isomorphism Theorem $KN/N \cong K/K \cap N$ and as the homomorphic image of $K$ is nilpotent by Theorem 2.5.8, the group $K/K \cap N$ is nilpotent. Then $KN/N$ is nilpotent. Now, consider the map

$$\varphi : A/A \cap N \mapsto AN/N$$

defined by

$$\varphi(a(A \cap N)) = aN, \quad \textit{where} \quad a \in A.$$

$\varphi$ is an isomorphism. We will observe that $\varphi(K(A \cap N)/A \cap N) = KN/N$. Take any element $k(A \cap N) \in K(A \cap N)/A \cap N$ where $k \in K$. Then $\varphi(k(A \cap N)) = kN \in KN/N$. Therefore $\varphi(K(A \cap N)/A \cap N) \subseteq KN/N$. Conversely, take any element $kN \in KN/N$. Then $kN = \varphi(k(A \cap N)) \in \varphi(K(A \cap N)/A \cap N)$. Hence, $KN/N \subseteq \varphi(K(A \cap N)/A \cap N)$ and so the equality holds. Also one may see that

$$\varphi(N_{A/A\cap N}(K(A \cap N)/A \cap N)) = N_{AN/N}(KN/N). \tag{3.1}$$

To show the equality (3.1) take any element $a(A \cap N) \in N_{A/A\cap N}(K(A \cap N)/A \cap N))$. By definition of the normalizer we have $K^a(A \cap N) = K(A \cap N)$. So

$$K^aN = K^a(A \cap N)N = K(A \cap N)N = KN$$

and

$$(KN/N)^{aN} = K^aN/N = KN/N.$$

That is $\varphi(a(A \cap N)) = aN \in N_{AN/N}(KN/N)$. Conversely, take any element $aN \in N_{AN/N}(KN/N)$. Then $K^aN = KN$ and by Dedekind identity, $K^a(A \cap N)/(A \cap N) = A \cap K^aN/A \cap N$. Therefore,

$$(K(A \cap N)/A \cap N))^{a(A\cap N)} = K^a(A \cap N)/(A \cap N) = A \cap K^aN/A \cap N = A \cap KN/A \cap N,$$

which is equal to

$$K(A \cap N)/(A \cap N).$$

So, $a(A \cap N) \in N_{A/A\cap N}(K(A \cap N)/A \cap N)$ and $aN = \varphi(a(A \cap N)) \in \varphi(N_{A/A\cap N}(K(A \cap N)/A \cap N))$. Therefore, the equality (1) holds. That is

$$N_{A/A\cap N}(K(A \cap N)/A \cap N) \cong N_{AN/N}(KN/N). \tag{3.2}$$

Note that $K$ is a Carter subgroup of the soluble group $A$ and $A \cap N \triangleleft A$. Thus, by Lemma 3.1.7 part d, $K(A \cap N)/A \cap N$ is a Carter subgroup of $A/A \cap N$, which implies that

$$N_{A/A\cap N}(K(A\cap N)/A\cap N) = K(A\cap N)/A\cap N \cong K/K\cap A\cap N = K/K\cap N \cong KN/N. \tag{3.3}$$

Thus by (3.2) and (3.3),

$$N_{AN/N}(KN/N) \cong KN/N$$

and $\left|N_{AN/N}(KN/N)\right| = |KN/N|$. So $KN/N \leq N_{AN/N}(KN/N)$ gives that

$$N_{AN/N}(KN/N) = KN/N.$$

Hence $KN/N$ is a Carter subgroup of $AN/N$. $\square$

### 3.2 Gaschütz Subgroups

**Definition 3.2.1** *[3]. A subgroup $H$ of a group $G$ is called a* **Gaschütz subgroup** *if $H$ satisfies the following two conditions:*

1) *$H$ is supersoluble.*

2) *if $H \leq H_1 < T \leq G$, then $|T : H_1|$ is not prime.*

**Lemma 3.2.2** 1) *In every soluble group, the Gaschütz subgroups exist and are mutually conjugate.*

2) *If $H \subseteq M \leq G$ and $H$ is a Gaschütz subgroup of $G$, then $H$ is a Gaschütz subgroup of $M$ .*

3) *Let $G$ be a group, let $A$ be a subgroup of $G$, and let $N$ be a normal subgroup of $G$. If $H$ is a Gaschütz subgroup of $A$, then $HN/N$ is a Gaschütz subgroup of $AN/N$.*

**Proof.**

1) See [[3], III, 4.24].

2) It is obvios that $H$ is supersoluble. Now assume that $H \leq H_1 < T \leq M$. Then

$$H \leq H_1 < T \leq M \leq G$$

implies that $H \leq H_1 < T \leq G$. Therefore $|T : H_1|$ is not prime. Hence $H$ is a Gaschütz subgroup of $M$.

3) $HN/N$ is supersoluble as by the Second Isomorphism Theorem it is isomorphic to the homomorphic image $H/H \cap N$ of the supersoluble group $H$, see Theorem 2.2.4 part(ii). Let

$$HN/N \leq H_1/N < T/N \leq AN/N.$$

By the Dedekind identity,

$$(A \cap H_1)N = AN \cap H_1 = H_1, \quad (A \cap T)N = AN \cap T = T.$$

Thus we get

$$1 \neq |T : H_1| = \frac{|T|}{|H_1|} = \frac{|(A \cap T)N|}{|(A \cap H_1)N|} = \frac{\frac{|A \cap T||N|}{|(A \cap T) \cap N|}}{\frac{|A \cap H_1||N|}{|(A \cap H_1) \cap N|}} = \frac{\frac{|A \cap T||N|}{|A \cap N|}}{\frac{|A \cap H_1||N|}{|A \cap N|}} = \frac{|A \cap T|}{|A \cap H_1|} = |A \cap T : A \cap H_1|$$

and

$$H \leq A \cap H_1 < A \cap T \leq A.$$

Since $H$ is a Gaschütz subgroup of $A$, it follows that $|A \cap T : A \cap H_1|$ is not prime. Thus,

$$|T : H_1| = |T/N : H_1/N|$$

is not prime. Therefore, $HN/N$ is a Gaschütz subgroup of $AN/N$.

□

### 3.3 Main Results

**Theorem 3.3.1** *[4]. Suppose that $G$ is a finite group, $A$ and $B$ are subgroups of $G$ and $AB^x = B^xA$ for all $x \in G$. Then the following hold:*
*If $G = AB^G = BA^G$, then $G = AB$.*

**Proof.** Let $G = AB^G = BA^G$. Proof by induction on $|G : A|$. Let $A$ be a normal subgroup of $G$, that is $A^g = A$ for any $g \in G$. So, $G = A^G B = AB$. If $A$ is not normal subgroup of $G$, then $N_G(A) < G$, so there is an element $y \in G$ such that $y \notin N_G(A)$. Here, $y \in G = AB^G$ then $y = ab^g$ for some $a \in A$, $b \in B$, $g \in G$. If $b^g \in N_G(A)$, then $y = ab^g \in N_G(A)$; a contradiction. So, $b^g \notin N_G(A)$ and therefore $A^{b^g} \neq A$. Set $A^* =< A, A^{b^g} >$ and $B^* = B^g$. We see that, $A < A^*$ as $A^{b^g} \neq A$. Hence, $|G : A^*| < |G : A|$. Also,

$$G = AB^G \leq A^*B^G = A^*(B^g)^G = A^*B^{*G}$$

gives that $G = A^*B^{*G}$. And

$$G = G^g = (BA^G)^g = B^g(A^G)^g = B^*A^G \leq B^*(A^*)^G$$

gives $G = B^*A^{*G}$. By induction hypothesis, $G = A^*B^*$. Then,

$$AB^g =< A, B^g >=< A, A^{b^g}, B^g >= A^*B^* = G.$$

Now, $g \in G = AB^g = Ag^{-1}Bg$. Thus, $g = a_1g^{-1}b_1g$ and so from here we can get $g = b_1a_1$. Hence, $G = AB^g = AB^{b_1a_1} = AB^{a_1}$. Taking $a_1^{-1}$ conjugate of both sides we get,

$$G^{a_1^{-1}} = A^{a_1^{-1}}B.$$

Thus, $G = AB$. □

The following three Lemmas are needed to prove the next theorems.

**Lemma 3.3.2** *[4]. If $A$ and $B$ are subgroups of a group $G$ such that $AB^x = B^xA$ for any $x \in G$, then the subgroup $A^B \cap B^A$ is subnormal in $G$.*

**Proof.** Assume that $A$ and $B$ are subgroups of the group $G$ such that $AB^x = B^xA$ for any $x \in G$. We use induction on the order of $G$. Let $G = AB = BA$, then

$$A^G \cap B^G = A^{AB} \cap B^{BA} = A^B \cap B^A.$$

As, $A^G \trianglelefteq G$ and $B^G \trianglelefteq G$, we have $A^B \cap B^A = A^G \cap B^G \trianglelefteq G$. Now, let $G \neq AB$, by Theorem 3.3.1, $G \neq AB^G$ or $G \neq BA^G$. Suppose that $AB^G < G$. Then by induction hypothesis, $A^B \cap B^A$ $sn$ $AB^G$. Also we have $A^B \cap B^A \leq B^A \leq B^G$. Thus,

$$A^B \cap B^A = (A^B \cap B^A) \cap B^G \ sn \ AB^G \cap B^G = B^G \trianglelefteq G.$$

Therefore, $A^B \cap B^A$ $sn$ $G$. □

**Lemma 3.3.3** *(Kazarin Theorem)[9]. If A and B are subgroups of a group G such that G = AB, A is nilpotent of odd order, and B is supersoluble, then G is soluble.*

**Theorem 3.3.4** *[4]. A join of finitely many subnormal soluble subgroups is always soluble.*

**Lemma 3.3.5** *If A is a soluble subnormal subgroup of a finite group G, then $A^G$ is soluble.*

**Proof.** Since $G$ is finite, there exist finitely many conjugates of $A$, say $A^{x_1}, A^{x_2}, \ldots, A^{x_k}$ for some $x_1, x_2, \ldots, x_k \in G$. Each $A^{x_i}$ is soluble as $A^{x_i} \cong A$ for all $i = 1, 2, \ldots, k$. Now the subgroup $A$ is subnormal in $G$ and so it has the series

$$A \trianglelefteq A_1 \trianglelefteq A_2 \cdots \trianglelefteq A_n = G.$$

Therefore,

$$A^{x_i} \trianglelefteq A_1^{x_i} \trianglelefteq A_2^{x_i} \cdots \trianglelefteq A_n^{x_i} = G.$$

That is $A^{x_i}$ is subnormal in $G$. Hence $A^G =< a^g : a \in A, g \in G >=< A^{x_1}, A^{x_2}, \ldots, A^{x_k} >$ is soluble by Theorem 3.3.4. □

**Theorem 3.3.6** *Let A and B be soluble subgroups of a group G and G = AB.*

1) *If every Carter subgroup of A commutes with every Carter subgroup of B, then the group G is soluble.*
2) *If the Carter subgroups of A are of odd order and every Carter subgroup of A commutes with every Gaschütz subgroup of B, then the group G is soluble.*

**Proof.** We will prove both assertions by induction on the order of the group.

1) Assume that every Carter subgroup of $A$ commutes with every Carter subgroup of $B$. Let $R(G) \neq 1$, and let $N$ be a minimal normal soluble subgroup of $G$. Then

$$GN/N = (AB)N/N = (AN/N)(BN/N),$$

and the subgroups

$$AN/N \cong A/A \cap N, \quad BN/N \cong B/B \cap N$$

are soluble. Let $C/N$ be a Carter subgroup of $AN/N$ and $K$ be a Carter subgroup of $A$. By Lemma 3.1.8, $KN/N$ is a Carter subgroup of $AN/N$. By Lemma 3.1.7 part(c), for some $a \in A$, $C/N = (KN/N)^{aN} = K^aN/N$. Then, $C = K^aN$. Similarly, if $E/N$ is a Carter subgroup of $BN/N$, then, by Lemma 3.1.7 part(c), we have $E = L^bN$, for some $b \in B$ and a Carter subgroup $L$ of $B$. Note that by Lemma 3.1.7 part(a), $K^a$ is a Carter subgroup of $A$ and $L^b$ is a Carter subgroup of $B$. So, by the assumption of the Theorem, $K^a$ and $L^b$ commute. Thus the subgroups $C/N = K^aN/N$ and $E/N = L^bN/N$ commute. By induction, the group $G/N$ is soluble and therefore the group $G$ is soluble by Theorem 2.1.15. Now suppose that $R(G) = 1$. Let $K$ be an arbitrary Carter subgroup of $A$ and let $x = ba$ be an arbitrary element in $G = AB = BA$ where $b \in B$ and $a \in A$. Let $L$ be an arbitrary Carter subgroup of $B$. Then by assumption any conjugate of $K$ in $A$ commutes with any conjugate of $L$ in $B$ and therefore

$$KL^x = KL^{ba} = (K^{a^{-1}}L^b)^a = (L^bK^{a^{-1}})^a = L^xK.$$

Set $D = K^L \cap L^K$. By Lemma 3.3.2, $D$ is a subnormal subgroup of $G$. Since $K^L \subseteq KL$ and $KL$ is soluble, by the Kegel-Wielandt Theorem (see Theorem 2.5.3), then $D$ is a soluble subnormal subgroup of $G$. By Lemma 3.3.5, $D^G$ is a soluble normal subgroup of $G$. Hence,

$$K^L \cap L^K = D \subseteq D^G \subseteq R(G) = 1.$$

Since $K^L = K^{(KL)}$ and $L^K = L^{(KL)}$ are normal in $KL$, it follows that

$$[K, L] \subseteq [K^L, L^K] \subseteq K^L \cap L^K = 1.$$

That is $K$ and $L$ commute element wise. Thus, every Carter subgroup of $A$ is contained in the centralizer of every Carter subgroup of $B$, and therefore $[K^A, L^B] = 1$. However, $K^A = A$ by Lemma 3.1.7 part(e).
Let $H = N_G(K^A) = N_G(A)$. Then,

$$L^B \subseteq C_G(A) \subseteq N_G(A) = H.$$

As $A$ is soluble and normal in $H$, $R(H) \neq 1$. Then, $H$ is a proper subgroup of $G$. By the Dedekind identity,

$$H = G \cap H = AB \cap H = A(B \cap H)$$

By Lemma 3.1.7 part(e), $L^B = B$. Thus, $B \subseteq H$. This implies that

$$H = A(H \cap B) = AB = G.$$

A contradiction.

2) Assume that the Carter subgroups of $A$ are of odd order and every Carter subgroup of $A$ commutes with every Gaschütz subgroup of $B$. Suppose that $R(G) \neq 1$, and let $N$ be a minimal normal soluble subgroup of $G$. Then

$$GN/N = (AB)N/N = (AN/N)(BN/N),$$

and the subgroups

$$AN/N \cong A/A \cap N, \quad BN/N \cong B/B \cap N$$

are soluble. Let $C/N$ be a Carter subgroup of of $AN/N$ of odd order and $K$ be a Carter subgroup of $A$ of odd order. By Lemma 3.1.8, $KN/N$ is a Carter subgroup of odd order of $AN/N$. By Lemma 3.1.7 part(c), there is an element $a \in A$, such that $C/N = (KN/N)^{aN} = K^aN/N$. Then, $C = K^aN$. Similarly, if $E/N$ is a Gaschütz subgroup of $BN/N$, then by Lemma 3.2.2 part(1), we have $E = L^bN$ for some $b \in B$ and a Gaschütz subgroup $L$ of $B$. Note that $K^a$ is a Carter subgroup of $A$ of odd order and $L^b$ is a Gaschütz subgroup of $B$ therefore by assumption of the Theorem $K^a$ and $L^b$ commute. Thus the subgroups $C/N = K^aN/N$ and $E/N = L^bN/N$ commute. By induction, the group $G/N$ is soluble and therefore the group $G$ is soluble by Theorem 2.1.15. Now suppose that $R(G) = 1$. Let $K$ be an arbitrary Carter subgroup of $A$ of odd order and Let $L$ be an arbitrary Gaschütz subgroup of $B$. Take any element $x = ba$ in $G = AB = BA$ where $b \in B$ and $a \in A$. Then by assumption any conjugate of $K$ in $A$ commutes with any conjugate of $L$ in $B$ and therefore

$$KL^x = KL^{ba} = (K^{a^{-1}} L^b)^a = (L^b K^{a^{-1}})^a = L^x K.$$

Set $D = K^L \cap L^K$. By Lemma 3.3.2, $D$ is a subnormal subgroup of $G$. Since $K^L \subseteq KL$ and $KL$ is soluble, by the Kazarin Theorem see Lemma 3.3.3, $D$ is a soluble subnormal subgroup of $G$. Therefore by Lemma 3.3.5, $D^G$ is a soluble normal subgroup of $G$. Hence,

$$K^L \cap L^K = D \subseteq D^G \subseteq R(G) = 1.$$

Since $K^L = K^{(KL)}$ and $L^K = L^{(KL)}$ are normal in $KL$, we have

$$[K, L] \subseteq [K^L, L^K] \subseteq K^L \cap L^K = 1.$$

That is $K$ and $L$ commute element wise. Thus, every Carter subgroup of $A$ is contained in the centralizer of every Gaschütz subgroup of $B$, and therefore $[K^A, L^B] = 1$. However, $K^A = A$ by Lemma 3.1.7 part(e).
Let $H = N_G(K^A) = N_G(A)$. Then

$$L^B \subseteq C_G(A) \subseteq N_G(A) = H.$$

As $A$ is soluble and normal in $H$, $R(H) \neq 1$. Then, $H$ is a proper subgroup of $G$ and by the Dedekind identity,

$$H = G \cap H = AB \cap H = A(B \cap H)$$

As $L \subseteq L^B \subseteq H$ and $L \subseteq B$ we get $L \subseteq H \cap B$. Then by Theorem 3.2.2 part(2), $L$ is a Gaschütz subgroub of $H \cap B$. Let $X$ be a Gaschütz subgroup of $H \cap B$. Thus by Lemma 3.2.2 part(1), $X$ and $L$ are conjugate in $H \cap B$ and $X$ is a Gaschütz subgroup of $B$. By assumption, $K$ and $X$ commute. Since $|H| < |G|$, $H$ satisfies the condition of the theorem and, by induction, $H$ is soluble. Now $HB = A(H \cap B)B = AB = G$. Since $L^B \subseteq H \cap B$, it follows that $L^G = L^{BH} \leq H$ which means $L^G$ is soluble. But, $L^G \subseteq R(G) = 1$. A contradiction.

□

**Theorem 3.3.7** *Let $A$ and $B$ be soluble subgroups of a group $G$ and $G = AB$.*

1) *If every Sylow subgroup of $A$ commutes with every Sylow subgroup of $B$, then the group $G$ is soluble.*

2) *If every Sylow subgroup of $A$ commutes with every Carter subgroup of $B$, then the group $G$ is soluble.*

3) *If the subgroup $A$ is of odd order and every Sylow subgroup of $A$ commutes with every Gaschütz subgroup of $B$, then the group $G$ is soluble.*

**Proof.** To prove all of the three parts we will use induction on the order of the group.

**Case 1**:

Suppose that $R(G) \neq 1$, and let $N$ be a minimal normal soluble subgroup of $G$. Then

$$GN/N = (AB)N/N = (AN/N)(BN/N),$$

and the subgroups

$$AN/N \cong A/A \cap N, \quad BN/N \cong B/B \cap N$$

are soluble. Let $p \in \pi(AN/N)$ and $X/N$ be a Sylow p-subgroup of $AN/N$, let $P$ be a Sylow p-subgroup of $A$. Then, $p$ does not divide $|A : P|$. Since $PN/N \cong P/P \cap N$, we get $PN/N$ is a p-group. Also,

$$|AN/N : PN/N| = |AN : PN| = |A(PN) : PN| = |A : A \cap PN|,$$

and $|A : P| = |A : A \cap PN| |A \cap PN : P|$. Since $p$ does not divide $|A : P|$, $p$ does not divide $|AN/N : PN/N|$. Therefore, $PN/N$ is a Sylow p-subgroup of $AN/N$. So by Theorem 2.3.7, $X/N$ and $PN/N$ are conjugate in $AN/N$. That is there exists $a \in A$ such that $X/N = P^a N/N$.

1) Assume that every Sylow subgroup of $A$ commutes with every Sylow subgroup of $B$. Let $q \in \pi(BN/N)$ and $Y/N$ be a Sylow q-subgroup of $BN/N$. Then, as showed above for any Sylow q-subgroup $Q$ of $B$, $QN/N$ is a Sylow q-subgroup of $BN/N$ and $Y/N = Q^b N/N$ for some $b \in B$. By the assumption of the Theorem, $P^a$ and $Q^b$ commute. Thus the subgroups $X/N = P^a N/N$ and $Y/N = Q^b N/N$

commute. By induction, the group $G/N$ is soluble and therefore the group $G$ is soluble by Theorem 2.1.15.

2) Assume that every Sylow subgroup of $A$ commutes with every Carter subgroup of $B$. Let $E/N$ be a Carter subgroup of $BN/N$ and let $L$ be a Carter subgroup of $B$. By Lemma 3.1.8, $LN/N$ is a Carter subgroup of $BN/N$ and also by Lemma 3.1.7 part(c), $E/N = (LN/N)^{bN} = L^b N/N$ for some $b \in B$. Then, $E = L^b N$ where $L^b$ is also a Carter subgroup of $B$. So, by the assumption of the Theorem, $P^a$ and $L^b$ commute. Thus the subgroups $X/N = P^a N/N$ and $E/N = L^b N/N$ commute. By induction, the group $G/N$ is soluble and therefore the group $G$ is soluble by Theorem 2.1.15.

3) Assume that the subgroup $A$ is of odd order and every Sylow subgroup of $A$ commutes with every Gaschütz subgroup of $B$. Let $E/N$ be a Gaschütz subgroup of $BN/N$ and let $L$ be a Gaschütz subgroup of $B$. Then, by Lemma 3.2.2 part(3), $LN/N$ is a Gaschütz subgroup of $BN/N$ and so by Lemma 3.2.2 part(1), $E/N$ and $LN/N$ are conjugate in $BN/N$. Thus, $E = L^b N$ for some $b \in B$. Note that $L^b$ is also Gaschütz subgroup of $B$. Therefore by assumption of the Theorem $P^a$ and $L^b$ commute. Thus the subgroups $X/N = P^a N/N$ and $E/N = L^b N/N$ commute. By induction, the group $G/N$ is soluble and therefore the group $G$ is soluble by Theorem 2.1.15.

**Case 2**:

Suppose that $R(G) = 1$. Let $P$ be a Sylow p-subgroup of $A$ and $L$ be a Sylow q-subgroup ( a Carter subgroup in (2) and a Gaschütz subgroup in (3)) of $B$. Take any element $x = ba$ in $G = AB = BA$ where $b \in B$ and $a \in A$. Then by the assumption any conjugate of $P$ in $A$ commutes with any conjugate of $L$ in $B$ and therefore

$$PL^x = PL^{ba} = (P^{a^{-1}} L^b)^a = (L^b P^{a^{-1}})^a = L^x P.$$

Set $D = P^L \cap L^P$. By Lemma 3.3.2, $D$ is a subnormal subgroup of $G$. Since $P^L \subseteq PL$ and $PL$ is soluble, in (1) and (2) by the Kegel-Wielandt Theorem (see Theorem 2.5.3) and in (3) by Kazarin Theorem (see Theorem 3.3.3), $D$ is a soluble subnormal subgroup of $G$. By Lemma 3.3.5, $D^G$ is a soluble normal subgroup of $G$. Hence,

$$P^L \cap L^P = D \subseteq D^G \subseteq R(G) = 1.$$

Since $P^L = P^{(PL)}$ and $L^P = L^{(PL)}$ are normal in $PL$, it follows that

$$[P, L] \subseteq [P^L, L^P] \subseteq P^L \cap L^P = 1.$$

That is $P$ and $L$ commute element wise.

1) Thus, every Sylow subgroup of $A$ is contained in the centralizer of every Sylow subgroup of $B$ and so $[P^A, L^B] = 1$. Set $H = N_G(P^A)$. Then $A \leq H$ and $L^B \subseteq C_G(P^A) \subseteq N_G(P^A) = H$. As, $P^A$ is soluble and normal in $H$, $R(H) \neq 1$ and so, $H$ is a proper subgroup of $G$. Let $|B| = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$ and $L_i$ be a Sylow $p_i$-subgroup of $B$. Then, $|L_i| = p_i^{k_i}$ and $L_i^B \subseteq C_G(P^A)$ for all $i$. As, $L_i^B \trianglelefteq B$ for all $i$, we get $L_1^B L_2^B \dots L_r^B \leq B$ and $p_i^{k_i}$ divides $\left|L_1^B L_2^B \dots L_r^B\right|$ for all $i$. So $|B| \leq \left|L_1^B L_2^B \dots L_r^B\right|$ and then $B = L_1^B L_2^B \dots L_r^B$. Hence $B \leq C_G(P^A)$. That is, $B \leq H$. By the Dedekind identity,

$$H = G \cap H = AB \cap H = A(B \cap H) = AB = G.$$

A contradiction.

2) Thus, every Sylow subgroup of $A$ is contained in the centralizer of every Carter subgroup of $B$ and so $[P^A, L^B] = 1$. Set $H = N_G(P^A)$. Then $A \leq H$ and $L^B \subseteq C_G(P^A) \subseteq N_G(P^A) = H$. As $P^A$ is soluble and normal in $H$, $R(H) \neq 1$ and so, $H$ is a proper subgroup of $G$. By the Dedekind identity,

$$H = G \cap H = AB \cap H = A(B \cap H)$$

By Lemma 3.1.7 part(e), $L^B = B$. So, $B \subseteq H$ and so

$$H = A(H \cap B) = AB = G.$$

A contradiction.

3) Thus, every Sylow subgroup of $A$ is contained in the centralizer of every Gaschütz subgroup of $B$. Therefore, $[P^A, L^B] = 1$. Set $H = N_G(P^A)$. Then, $A \leq H$ and

$$L^B \subseteq C_G(P^A) \subseteq N_G(P^A) = H.$$

As $P^A$ is soluble and normal in $H$, $R(H) \neq 1$. Then, $H$ is a proper subgroup of $G$ also by the Dedekind identity,

$$H = G \cap H = AB \cap H = A(B \cap H).$$

As $L \subseteq L^B \subseteq H$ and $L \subseteq B$ we get $L \subseteq H \cap B$. Then by Theorem 3.2.2 part(2), $L$ is a Gaschütz subgroub of $H \cap B$. Let $X$ be a Gaschütz subgroup of $H \cap B$. Thus by Lemma 3.2.2 part(1), $X$ and $L$ are conjugate in $H \cap B$ and $X$ is a Gaschütz subgroup of $B$. By assumption, $P^A$ and $X$ commute. Since $|H| < |G|$, $H$ satisfies the condition of the theorem and, by induction, $H$ is soluble. Now $HB = A(H \cap B)B = AB = G$. Since $L^B \subseteq H \cap B$, it follows that $L^G = L^{BH} \leq H$ which means $L^G$ is soluble. But, $L^G \subseteq R(G) = 1$. A contradiction.

□

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